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ELEMENTARY THEOREMS

RELATING TO

DETERMINANTS.

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PREFACE.

THE variety of problems to which the Theory of Determinants has recently been applied renders it desirable that this branch of analysis should be made generally accessible. But although the principal theorems are familiar to the more advanced mathematicians, there has hitherto been no elementary work upon the subject, to which reference can be readily made by the student.

The Theory is neither lengthy nor intricate, being in fact little else than a method of arrangement, by means of which the results of certain long algebraic processes may be discovered without actually effecting the operations; and indeed, with the exception of a few theorems relating to the addition, multiplication, &c. of determinants, it may be said to consist entirely in its application. Like all similar calculi, it may be carried out into very numerous details; but although this has not been attempted in the present investigations, the principal modifications of form and varieties of combination have been noticed, and the theorems throughout illustrated by examples. The reader will be thereby enabled generally to apply the processes whenever opportunity occurs, and to comprehend any new theorems which may hereafter be proposed. The demonstrations here offered are principally original, although perhaps not different from such as may have occurred to others who have paid attention to the subject.

The functions which are the subject of the present paper, or cases of them more or less general, have for many years been an object of interest to mathematicians; in fact so long ago as the year 1750, Cramer, in his *Introduction à l'Analyse des lignes Courbes* (Appendix), has exhibited the determinants arising from linear equations in the case of two or three variables, and has indicated the law according to which they would be formed in the case of a greater number. In the *Histoire de l'Academie Royale des Sciences, Année* 1764 (published in 1767) Bézout has investigated the degree of the equation resulting from the elimination of unknown quantities from a given system of equations, and has at the same time noticed several cases of determinants, without however entering upon the general law of formation, or the properties of these functions. The *Hist. de l'Académie, An.* 1772, Part II. (published in 1776) contains papers by Laplace and Vandermonde relating to determinants of the second, third, fourth, &c. order. The former, in discussing a system of simultaneous differential equations, has given the law of formation, and shown that when two horizontal or vertical rows (according to the notation of the present work) are interchanged, the sign of the determinant is changed. Vandermonde's paper is upon elimination, and considering the period at which it was written, is remarkable for its elegance; the notation, which is worth noticing, is as follows; the system of quantities being thus represented,

$$\begin{matrix} {}^1 1 & {}^1 2 & .. & {}^1 n \\ {}^2 1 & {}^2 2 & .. & {}^2 n \\ . & . & .. & . \\ {}^n 1 & {}^n 2 & .. & {}^n n \end{matrix}$$

a determinant of the nth order is written thus,

$$\frac{1 \mid 2 \mid .. \mid n}{1 \mid 2 \mid .. \mid n}$$

so that

$$\frac{1 \mid 2}{1 \mid 2} = {}^1 1 \; {}^2 2 - {}^2 1 \; {}^1 2$$

and so on for other orders.

In the *Memoires de l'Académie de Berlin*, 1773, Lagrange has demonstrated that the square of a determinant of the third order is itself a determinant; these formulæ he applies to the establishment of theorems relating to triangular pyramids, and to the problem of the rotation of a solid body. Subsequently to this, Gauss, in his *Disquisitiones Arithmeticæ*, has shown (Section V. Nos. 159 and 270) that the product of two determinants is itself a determinant in the cases of the second and third orders. The whole of this section, which forms a large portion of the work, is devoted to these functions. The case of determinants of the second order arising from quadratic functions of two variables, i.e. of the form $b^2 - ac$, or, adopting his notation, (a, b, c), is very completely discussed. And besides the theorem above noticed, the following problem, which has some connexion with determinants of determinants, is solved: "Given any three whole numbers a, a', a'', (which are not all $= 0$), to find six others, B, B′, B″, C, C′, C″, such that $B'C'' - B''C' = a$, $B''C - BC'' = a'$, $BC' - B'C = a''$." This mathematician appears to have also introduced the term Determinant.

In 1812 Binet published a memoire upon this subject, and established all the principal theorems for determinants of the second, third, and fourth orders; and has further applied his formulæ to the discussion of rhomboids, surfaces of the second order, and properties of solid bodies. See *Journal de l'Ecole Polytechnique, tome* IX. *cahier* 16. The next volume of this series contains a paper by Cauchy, written at the same time, on functions which only change sign when the variables which they contain are transposed. The second part of this paper refers immediately to determinants, and contains a large number of very general theorems. Amongst them is noticed a property of a class of functions closely connected with determinants, first given, so far as I am aware, by Vandermonde; if in the development of the expression

$$a_1 a_2 \cdot\cdot a_n (a_2 - a_1)(a_3 - a_1)(a_n - a_1)(a_3 - a_2) \cdot\cdot (a_n - a_2)(a_n - a_{n-1})$$

the indices be replaced by a second series of suffixes, the result will be the determinant

$$S(\pm a_{1,1} \; a_{2,2} \cdot\cdot a_{n,n})$$

Several papers appeared subsequently from time to time upon various points connected with the subject; but by far the most complete are two by M. Jacobi (Crelle, tom. XXII.) *De formâ et proprietatibus Determinantium*, and *De Determinantibus Functionalibus.* In the same Journal (tom. XXXII. and XXXVIII.) there are two memoires by Mr. Cayley, *Sur les Determinantes Gauches*, which expression has been rendered *Skew Determinants*, the term being adopted from the corresponding translation in Geometry.

References will be found in the course of this work to other papers in which Determinants have been employed, all of which may be consulted with advantage. Besides these, there may be mentioned the following; "On the Theory of Elimination," by Mr. Cayley, *Camb. & Dub. Math. Journal*, vol. III.; "On a new Class of Theorems, &c." by Mr. Sylvester, *Phil. Mag.* vol. XXXVII. ; "Extraits de lettres de M. Ch. Hermite, à M. C. G. J. Jacobi, sur differents objects de la théorie des nombres," *Crelle*, tom. XL.

Besides that which is here discussed, there is another very interesting and apparently important theory lately proposed by Mr. Cayley relating to functions, which he calls *Hyperdeterminants.* The general question therein proposed is, "'To find all the derivatives of any number of functions which have the property of preserving their forms unaltered after any linear transformations of the variables.' By derivative is to be understood a function deduced in any manner whatever from the given function, and by hyperdeterminant derivative, or simply hyperdeterminant, those derivatives which have the property above enunciated." Of this nothing has been here said, but those who are desirous of pursuing the subject will find the principles of it laid down in two papers, *Camb. Math. Journal*, vol. iv., and *Camb. and Dub. Math. Journal*, vol. i., or in *Crelle*, tom. xxx.

ELEMENTARY THEOREMS

RELATING TO

DETERMINANTS.

---

§ I.—*Introduction.*

Before entering upon the general Theory of Determinants, it will be desirable to notice a few well known formulæ, which are in fact themselves determinants, and which exhibit particular cases of the more general theorems hereafter to be established. It will thus be seen that the present theory involves no new principle, and that a determinant is nothing else than a simplified method of notation, by means of which transformations involving long processes of multiplication may be effected at sight. As these functions generally occur either in direct connexion with linear equations, or in such a way that they may be resolved into a system of linear equations, it will be simplest to introduce them as so connected. Perhaps no more familiar illustration of the theory can be found than that afforded by the equations for determining the centre and principal axes of a plane curve or of a surface of the second order. The determinants so arising are not indeed the most general of their class, but as this fact does not materially affect their forms or relations, the illustration will be as good as if their character was perfectly general.

The equations for determining the centre and principal axes of the plane curve whose equation is

$$Ax^2 + Cy^2 + 2(Bxy + Dx + Ey) = K$$

are, as is well known,

$$\begin{gathered} A\alpha + B\beta = D \\ B\alpha + C\beta = E \\ (A - \theta)l + Bm = \\ Bl + (C - \theta)m = 0 \end{gathered}$$

where α β are the co-ordinates of the centre l, m, the direction-cosines of one of the principal axes, and θ a root of the quadratic formed by the elimination of l and m from the two latter equations. The quadratic in question is in fact the following,

$$(A-\theta)(C-\theta)-B^2=0,$$

and the function forming the left-hand side of this equation is a *determinant;* and, being of the second order in the quantities involved, is called a *determinant of the second order*.

The quantities involved, $A-\theta$, $C-\theta$, B, are called *the constituents of the determinant;* but, according to the method of the following pages, this and other like functions will be written not as above, but in the following manner,

$$\begin{vmatrix} A-\theta & B \\ B & C-\theta \end{vmatrix}$$

and whenever such an expression occurs it will be understood that the four constituents are to be multiplied together two and two diagonally as regards the square in which they are arranged, and that the difference of the two products will be the function indicated. It is also to be borne in mind that the product involving the quantity at the upper left-hand corner of the square is to be taken positively, and the other of course negatively.

Again, from what has now been said, it is not difficult to see that the determinant

$$\begin{vmatrix} A & B \\ B & C \end{vmatrix}$$

may be considered as that function which when equated to zero would express the result of the elimination of l, m, from the two linear equations

$$Al+Bm=0$$
$$Bl+Cm=0$$

or it may be considered as expressing the common denominator of the values of α and β deduced from the equations

$$A\alpha+B\beta=D$$
$$B\alpha+C\beta=E$$

Two other determinants occur in the solution of these equations, which it may be worth the reader's while to verify; they are as follows,

$$\begin{vmatrix} A & B \\ B & C \end{vmatrix} \alpha = \begin{vmatrix} D & B \\ E & C \end{vmatrix} \qquad \begin{vmatrix} A & B \\ B & C \end{vmatrix} \beta = - \begin{vmatrix} D & A \\ E & B \end{vmatrix}$$

As a further exercise these latter equations may be solved with respect to α and β, and by dividing out the common factor

$$\begin{vmatrix} A & B \\ B & C \end{vmatrix}$$

the original equations reproduced. The following equations may also be verified,

$$\begin{vmatrix} A & B \\ B & C \end{vmatrix} = - \begin{vmatrix} B & A \\ C & B \end{vmatrix} = - \begin{vmatrix} B & C \\ A & B \end{vmatrix} = \begin{vmatrix} C & B \\ B & A \end{vmatrix}$$

so that in determinants of the second order, by an interchange of two horizontal or vertical rows the sign of the determinant is changed; and by two such interchanges the original sign is of course restored.

Again,

$$\begin{vmatrix} A & B \\ A & B \end{vmatrix} = 0 \qquad \begin{vmatrix} A & A \\ B & B \end{vmatrix} = 0$$

and consequently if either two vertical or two horizontal rows become equal, the determinant vanishes.

Again,

$$\begin{vmatrix} A + A' & B + B' \\ B + B' & C + C' \end{vmatrix} = \begin{vmatrix} A & B \\ B & C \end{vmatrix} + \begin{vmatrix} A & B' \\ B & C' \end{vmatrix} + \begin{vmatrix} A' & B \\ B' & C \end{vmatrix} + \begin{vmatrix} A' & B' \\ B' & C' \end{vmatrix}$$

i. e. the determinant, each of whose constituents is the sum of two others, is equal to the sum of the four determinants formed by taking the eight constituents four and four together, each constituent retaining its position in the square. As a corollary the following may be noticed:

$$\begin{vmatrix} A + B & B \\ B + C & C \end{vmatrix} = \begin{vmatrix} A & B \\ B & C \end{vmatrix}$$

It may also be shown that

$$\begin{vmatrix} A & B \\ C & B \end{vmatrix} = \begin{vmatrix} A & 1 \\ C & 1 \end{vmatrix} B, \qquad \begin{vmatrix} A^2 + B^2 & AB + BC \\ AB + BC & B^2 + C^2 \end{vmatrix} = \begin{vmatrix} A & B \\ B & C \end{vmatrix}^2$$

Returning to the determinant arising from the equations for determining the principal axes of the plane curve, it will be found by actual multiplication that

$$AC - B^2 - (A + C)\,\theta + \theta^2 = 0$$

which may also be written thus,

$$\begin{vmatrix} A & B \\ B & C \end{vmatrix} - (A + C)\theta + \theta^2 = 0$$

so that the term independent of θ is the determinant formed by putting $\theta = 0$ in the given determinant; the coefficient of θ is the sum of the quantities lying in the same diagonal with θ, and the coefficient of θ^2 is unity. The coefficients of the powers of θ are of course alternately positive and negative.

Proceeding to three dimensions, the centre and principal axes of a surface of the second order will be determined by the equations

$$\begin{aligned} A\alpha + H\beta + G\gamma &= L \\ H\alpha + B\beta + F\gamma &= M \\ G\alpha + F\beta + C\gamma &= N \end{aligned}$$

$$\begin{aligned} (A - \theta)\, l + Hm + Gn &= 0 \\ Hl + (B - \theta)\, m + Fn &= 0 \\ Gl + Fm + (C - \theta)\, n &= 0 \end{aligned}$$

Analogously to the case of two dimensions, the result of the elimination of l, m, n from the latter system will be thus expressed,

$$\begin{vmatrix} A-\theta & H & G \\ H & B-\theta & F \\ G & F & C-\theta \end{vmatrix} = 0$$

and the common denominator of the values of α, β, γ deduced from the first system, thus,

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

but by actual elimination it would be found that this expression is equivalent to

$$ABC - AF^2 - BG^2 - CH^2 + 2FGH$$

or, by what has been said above respecting determinants of the second order,

$$= A \begin{vmatrix} B & F \\ F & C \end{vmatrix} + H \begin{vmatrix} F & H \\ C & G \end{vmatrix} + G \begin{vmatrix} H & B \\ G & F \end{vmatrix}$$

which gives the law of the formation of determinants of the third order from those of the second; and they are in fact the sum of the products of each quantity in the top horizontal row and the determinants formed by the four quantities which lie two and two in each of the lower horizontal rows, and two and two in each of the vertical rows in which the quantity first chosen does not lie; the vertical rows are to be taken in the usual cyclic order, viz. (1) the second and third, (2) the third and first, (3) the first and second.

The theorems corresponding to those noticed in two dimensions may be verified without difficulty, as also may the following, which will not be found laborious if, as is probable, the reader is tolerably familiar with the usual formulæ in algebraical geometry.

$$\begin{vmatrix} \begin{vmatrix} C & G \\ G & A \end{vmatrix} & \begin{vmatrix} G & A \\ F & H \end{vmatrix} \\ \begin{vmatrix} G & A \\ F & H \end{vmatrix} & \begin{vmatrix} A & H \\ H & B \end{vmatrix} \end{vmatrix} = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} A$$

In this expression it is to be observed that the constituents of the determinant are themselves determinants; and the equation may be verified by developing both the determinant itself and the determinants which form its constituents; the order in which the developments are effected is of course arbitrary.

$$\begin{vmatrix} \begin{vmatrix} B & F \\ F & C \end{vmatrix} & \begin{vmatrix} F & C \\ H & G \end{vmatrix} & \begin{vmatrix} H & B \\ G & F \end{vmatrix} \\ \begin{vmatrix} F & C \\ H & G \end{vmatrix} & \begin{vmatrix} C & G \\ G & A \end{vmatrix} & \begin{vmatrix} G & A \\ F & H \end{vmatrix} \\ \begin{vmatrix} H & B \\ G & F \end{vmatrix} & \begin{vmatrix} G & A \\ F & H \end{vmatrix} & \begin{vmatrix} A & H \\ H & B \end{vmatrix} \end{vmatrix} = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}^2$$

This expression is again a determinant whose constituents are themselves determinants, and the equation may be verified in a variety of ways. The constituent determinants may be first developed, and afterwards the determinant whose constituents they are, or vice versâ, first the determinant and then the constituent determinants, and in either case the result must be compared with the development of the expression on the right-hand side of the equation. It will however be less laborious, and more in accordance with the methods hereafter laid down, to employ the formulæ already established, and thereby avoid a complete development of the expressions. We should then proceed as follows; in the same way that the preceding equation was verified, the following may be also established,

$$\begin{vmatrix} \begin{vmatrix} G & A \\ F & H \end{vmatrix} & \begin{vmatrix} F & C \\ H & G \end{vmatrix} \\ \begin{vmatrix} A & H \\ H & B \end{vmatrix} & \begin{vmatrix} H & B \\ G & F \end{vmatrix} \end{vmatrix} = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} H \qquad \begin{vmatrix} \begin{vmatrix} F & C \\ H & G \end{vmatrix} & \begin{vmatrix} C & G \\ G & A \end{vmatrix} \\ \begin{vmatrix} H & B \\ G & F \end{vmatrix} & \begin{vmatrix} G & A \\ F & H \end{vmatrix} \end{vmatrix} = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} G$$

and consequently the determinant to be developed may be thus expressed,

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} \left\{ A \begin{vmatrix} B & F \\ F & C \end{vmatrix} + H \begin{vmatrix} F & C \\ H & G \end{vmatrix} + G \begin{vmatrix} H & B \\ G & F \end{vmatrix} \right\}$$

but the expression within the brackets is, as was before shown, equal to the determinant outside the brackets; so that the whole expression is equal to the square of that determinant, as was to be proved.

$$\begin{vmatrix} A^2 + H^2 + G^2 & (A + B)H + FG & (C + A)G + HF \\ (A + B)H + FG & H^2 + B^2 + F^2 & (B + C)F + GH \\ (C + A)G + HF & (B + C)F + GH & G^2 + F^2 + C^2 \end{vmatrix} = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}^2$$

The point in this expression to be particularly remarked is, that each constituent of the determinant on the left-hand side of the equation is the sum of the constituents which lie in one of the horizontal rows of the determinant on the right-hand side multiplied respectively by the corresponding constituents in either the same or another horizontal row. Thus for the first vertical row on the left-hand side, there is formed, (1) the sum of A, H, G multiplied respectively by A, H, G; (2) the sum of A, H, G multiplied respectively by H, B, F; (3) the sum of A, H, G multiplied respectively by G, F, C; and similarly for all the other vertical rows.

And the determinant involving θ may be written thus,

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} - \left\{ \begin{vmatrix} B & F \\ F & C \end{vmatrix} + \begin{vmatrix} C & G \\ G & A \end{vmatrix} + \begin{vmatrix} A & H \\ H & B \end{vmatrix} \right\} \theta + (A + B + C)\theta^2 - \theta^3 = 0$$

So that the term independent of θ is the determinant of the third order, formed by putting $\theta = 0$ in the given determinant; the coefficient of θ is the sum of the determinants of the second order formed by omitting in turn the horizontal and vertical rows passing through the quantities, A, B, C; the coefficient of θ^2 is the sum of A, B, C, i.e. of the quantities lying upon the same diagonal with θ in the given determinant; and the coefficient of θ^3 is unity. The coefficients of the powers of θ are, as in the case of two dimensions, alternately positive and negative.

Another well-known instance of formulæ which give rise to determinants is that which occurs in the transformation of co-ordinates, especially when the transformation takes place in three dimensions. Thus to pass from one set of axes (x, y, z) to another (ξ, η, ζ), we have the equations

$$\begin{aligned} \xi &= lx + my + nz \\ \eta &= l'x + m'y + n'z \\ \zeta &= l''x + m''y + n''z \end{aligned}$$

and conversely to pass from the system (ξ, η, ζ) to (x, y, z), we should have the inverse system, which may be thus written;

$$\nabla x = \begin{vmatrix} \xi & m & n \\ \eta & m' & n' \\ \zeta & m'' & n'' \end{vmatrix}, \qquad \nabla y = \begin{vmatrix} \xi & n & l \\ \eta & n' & l' \\ \zeta & n'' & l'' \end{vmatrix}, \qquad \nabla z = \begin{vmatrix} \xi & l & m \\ \eta & l' & m' \\ \zeta & l'' & m'' \end{vmatrix}$$

where

$$\nabla = \begin{vmatrix} l & m & n \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix}$$

if both systems of axes are rectangular, the usual expression becomes

$$\nabla = \pm 1$$

It may further be observed that in all these determinants the same result will be obtained by developing according to the vertical instead of the horizontal rows. Thus the first stage of the development of ∇ may be either

$$l \begin{vmatrix} m' & n' \\ m'' & n'' \end{vmatrix} + m \begin{vmatrix} n' & l' \\ n'' & l'' \end{vmatrix} + n \begin{vmatrix} l' & m' \\ l'' & m'' \end{vmatrix}$$

or

$$l \begin{vmatrix} m' & n' \\ m'' & n'' \end{vmatrix} + l' \begin{vmatrix} m'' & n'' \\ m & n \end{vmatrix} + l'' \begin{vmatrix} m & n \\ m' & n' \end{vmatrix}$$

As a last example it may be noticed that the ordinary relations between the direction-cosines of two rectangular sets of axes may be expressed as determinants; thus the equations

$$ll' + mm' + nn' = 0 \qquad l^2 + m^2 + n^2 = 1$$

may be written as follows,

$$\begin{vmatrix} 1 & 0 & 0 & l \\ 0 & 1 & 0 & m \\ 0 & 0 & 1 & n \\ l' & m' & n' & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 0 & 0 & l \\ 0 & 1 & 0 & m \\ 0 & 0 & 1 & n \\ l & m & n & 0 \end{vmatrix} = -1, \text{ or } \begin{vmatrix} 1 & 0 & 0 & l \\ 0 & 1 & 0 & m \\ 0 & 0 & 1 & n \\ l & m & n & 1 \end{vmatrix} = 0$$

The form of these determinants, which are of the fourth order, might be discovered by eliminating four unknown quantities from four homogeneous linear equations; but as this is a rather long process, it may be assumed, as will hereafter be proved, that a determinant of the fourth order is formed by multiplying the constituents of the top horizontal row respectively by the determinants of the third order formed from the three remaining horizontal rows, three out of the four vertical rows being successively selected in the usual cyclic order; the products so formed are to be taken alternately positive and negative, as in the case of determinants of the second order.

By these exercises, and any others which may suggest themselves, the reader will have become sufficiently acquainted with the nature and object of the method to enter upon the general theory, to which we now proceed.

---

§ II.—*On the Formation of Determinants.*

In order to designate a certain system of objects, quantitative or other, it is usual to employ either different letters, such as

$$a, \ b, \ c, \ . \ .$$

or the same letter with accents or suffixes, as

$$k, \ k', \ k'', \ . \ .$$
$$k, \ k_1, \ k_2, \ . \ .$$

a second suffix being introduced when necessary, as

$$\begin{matrix} k_{1,1}, & k_{1,2}, & . \ . \\ k_{2,1}, & k_{2,2}, & . \ . \\ . & . & . \ . \end{matrix}$$

this notation being especially useful when the number of letters is of the form mn, since they may then be arranged in m vertical and n horizontal, or n vertical and m horizontal rows; in the case when m and n are equal the letters may of course be arranged in a square. It is, however, more simple and not less general to write down only the suffixes, omitting the letters to which they are supposed to refer; so that a system of n^2 letters will be thus expressed,

$$\begin{matrix} (1,1), & (1,2), & \ldots & (1,n), \\ (2,1), & (2,2), & \ldots & (2,n), \\ \cdot & \cdot & \cdot\;\cdot & \cdot \\ (n,1), & (n,2), & \ldots & (n,n). \end{matrix}$$

These symbols are perfectly general, and indicate the position in some primary arrangement to which they severally belong. Thus, any symbol (p,q) in such a system would be that which originally stood in the pth horizontal and qth vertical row, counting from the top and the left hand respectively. It must, however, be carefully borne in mind that the numbers employed in these symbols have nothing whatever to do with the nature of the quantity or operation whose position they determine, and that consequently the symbols (p,q) and (q,p) have in general no relation or connexion with one another.

These symbols will form the constituents of the determinants in the following investigations, and their natural order of arrangement will be that of the table given above. It will be noticed that the symbols $(1,1)$, $(2,2)$, . . will then form the diagonal row corresponding to that of A,B, . ., in the cases noticed in the first section, and that to pass from a horizontal to a vertical row, or vice versâ, it is only necessary to interchange the two numbers in each of the constituents, i. e. to write (q,p) for (p,q), or (p,q) for (q,p). In the equations for determining the centre and principal axes of a plane curve, $n=2$; in the corresponding case with respect to surfaces of the second order, $n=3$, and in both cases

$$(p,q)=(q,p)$$

i. e.

$$(2,3)=(3,2),\ (3,1)=(1,3),\ (1,2)=(2,1)$$

while in the equations for the transformation of co-ordinates these latter conditions are not fulfilled; but as the nature of the arrangement is perfectly obvious it is needless to give further explanation.

A determinant may now be defined by the following equations, each of which is comprised in that which succeeds it. The general expression for a determinant of the *n*th order is given below; but those of the orders, 1, 2, 3, 4, . . with their corresponding development, have been written down first for the sake of clearness.

$$\begin{vmatrix}(1,1)\end{vmatrix} = (1,1)$$

$$\begin{vmatrix}(1,1)(1,2)\\(2,1)(2,2)\end{vmatrix} = (1,1)\begin{vmatrix}(2,2)\end{vmatrix} - (1,2)\begin{vmatrix}(2,1)\end{vmatrix} = (1,1)(2,2)-(1,2)(2,1)$$

$$\begin{vmatrix}(1,1)(1,2)(1,3)\\(2,1)(2,2)(2,3)\\(3,1)(3,2)(3,3)\end{vmatrix} = (1,1)\begin{vmatrix}(2,2)(2,3)\\(3,2)(3,3)\end{vmatrix} + (1,2)\begin{vmatrix}(2,3)(2,1)\\(3,3)(3,1)\end{vmatrix} + (1,3)\begin{vmatrix}(2,1)(2,2)\\(3,1)(3,2)\end{vmatrix}$$

$$\begin{aligned}&= (1,1)(2,2)(3,3) - (1,1)(3,2)(2,3) + (1,2)(2,3)(3,1)\\&\quad - (1,2)(3,3)(2,1) + (1,3)(2,1)(3,2) - (1,3)(3,1)(2,2)\end{aligned}$$

$$\begin{vmatrix}(1,1)(1,2)(1,3)(1,4)\\(2,1)(2,2)(2,3)(2,4)\\(3,1)(3,2)(3,3)(3,4)\\(4,1)(4,2)(4,3)(4,4)\end{vmatrix} = (1,1)\begin{vmatrix}(2,2)(2,3)(2,4)\\(3,2)(3,3)(3,4)\\(4,2)(4,3)(4,4)\end{vmatrix} - (1,2)\begin{vmatrix}(2,3)(2,4)(2,1)\\(3,3)(3,4)(3,1)\\(4,3)(4,4)(4,1)\end{vmatrix}$$

$$+ (1,3)\begin{vmatrix}(2,4)(2,1)(2,2)\\(3,4)(3,1)(3,2)\\(4,4)(4,1)(4,2)\end{vmatrix} - (1,4)\begin{vmatrix}(2,1)(2,2)(2,3)\\(3,1)(3,2)(3,3)\\(4,1)(4,2)(4,3)\end{vmatrix}$$

$$\begin{aligned}&= (1,1)(2,2)(3,3)(4,4) - (1,1)(2,2)(4,3)(3,4) + (1,1)(3,2)(4,3)(2,4)\\&\quad - (1,1)(3,2)(2,3)(4,4) + (1,1)(4,2)(2,3)(3,4) - (1,1)(4,2)(3,3)(2,4)\\&\quad - (1,2)(2,3)(3,4)(4,1) + (1,2)(2,3)(4,4)(3,1) - (1,2)(3,3)(4,4)(2,1)\\&\quad + (1,2)(3,3)(2,4)(4,1) - (1,2)(4,3)(2,4)(3,1) + (1,2)(4,3)(3,4)(2,1)\\&\quad + (1,3)(2,4)(3,1)(4,2) - (1,3)(2,4)(4,1)(3,2) + (1,3)(3,4)(4,1)(2,2)\\&\quad - (1,3)(3,4)(2,1)(4,2) + (1,3)(4,4)(2,1)(3,2) - (1,3)(4,4)(3,1)(2,2)\\&\quad - (1,4)(2,1)(3,2)(4,3) + (1,4)(2,1)(4,2)(3,3) - (1,4)(3,1)(4,2)(2,3)\\&\quad + (1,4)(3,1)(2,2)(4,3) - (1,4)(4,1)(2,2)(3,3) + (1,4)(4,1)(3,2)(2,3)\end{aligned}$$

The law of formation of these functions will be sufficiently obvious, if it be noticed that when the number of rows (horizontal or vertical) is odd the terms in the first stage of the development are all positive, and when the number is even the terms are alternately positive and negative. It would be found on trial that this determinant of the fourth order is the common denominator in the expressions for four unknown quantities determined by four linear equations whose coefficients are the constituents of the determinant; or, that the same determinant, when equated to zero, is the result of the elimination of the same unknown quantities from the same equations, when their second members vanish.

The construction of determinants of the *n*th order, and their connexion with linear equations, are precisely the same as those of the particular cases above noticed. This may be shown either by forming the common denominator in the solution of a system of linear equations having second members, or, which is the same thing, by finding the condition that those equations may co-exist when their second members vanish; a process which would show that the functions so constructed are formed according to the same law as the determinants of the orders 1, 2, . . ., given above; or, again, since the law by which determinants of the orders 1, 2, . . are constructed is *sufficient* for the formation of a determinant of the order *n*, we may proceed in the inverse order, and construct a function of the same kind as those of the orders 1, 2, . . and then prove that it is identical with the common denominator, or the result of the elimination from the above-mentioned system of equations. Adopting the latter course, the function in question will be written thus,

$$\begin{vmatrix} (1,1)(1,2) \cdot\cdot (1,n) \\ (2,1)(2,2) \cdot\cdot (2,n) \\ . \quad . \quad .. \quad . \\ (n,1)(n,2) \cdot\cdot (n,n) \end{vmatrix}$$

the law of formation being as follows,

$$\nabla = (1,1) \begin{vmatrix} (2,2)(2,3) \cdot\cdot (2,n) \\ (3,2)(3,3) \cdot\cdot (3,n) \\ . \quad . \quad .. \quad . \\ (n,2)(n,3) \cdot\cdot (n,n) \end{vmatrix} \pm (1,2) \begin{vmatrix} (2,3)(2,4) \cdot\cdot (2,1) \\ (3,3)(3,4) \cdot\cdot (3,1) \\ . \quad . \quad .. \quad . \\ (n,3)(n,4) \cdot\cdot (n,n) \end{vmatrix}$$

$$+ \cdot\cdot \pm (1,n) \begin{vmatrix} (2,1)(2,2) \cdot\cdot (2,n-1) \\ (3,1)(3,2) \cdot\cdot (3,n-1) \\ . \quad . \quad .. \quad . \\ (n,1)(n,2) \cdot\cdot (n,n-1) \end{vmatrix}$$

the upper signs being taken when the number of rows (horizontal or vertical) is odd, and the lower when it is even.

It is easily seen by the law of its formation, that a determinant is the sum of a series of homogeneous products, and M. Jacobi and others have in consequence adopted the following notation,

$$\nabla = \Sigma \pm (1,1)\,(2,2) \ldots (n,n)$$

so that by the right-hand side of this equation is indicated the sum of terms formed by all possible interchange of the first and second members of the binary combinations, $(1,1)$, $(2,2)$, . . (n,n), subject to the condition that in each product all the first members shall be different, and likewise all the second members; the sign of each product being $+$ or $-$, according as it is deducible from

$$(1,1)\ (2,2) \cdot\cdot (n,n)$$

by an even or odd number of interchanges of the first (or second) members. This will become apparent if another step in the development of ∇ be effected; the first two terms of the series may then be written

$$\{(1,1)\ (2,2)-(1,2)\ (2,1)\} \begin{vmatrix} (3,3) & (3,4) & \cdot\cdot & (3,n) \\ (4,3) & (4,4) & \cdot\cdot & (4,n) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (n,3) & (n,4) & \cdot\cdot & (n,n) \end{vmatrix}$$

So that the single interchange of 1 and 2, producing the product $(1,2)\ (2,1)$ from $(1,1)\ (2,2)$, gives rise to one change of sign; and each subsequent interchange will similarly produce one change. Moreover, since each term will contain one (and only one) constituent from each horizontal row, and also one (and only one) from each vertical row, it easily follows that

Theorem I. *If the whole of a vertical or horizontal row be multiplied by the same quantity, the determinant is multiplied by that quantity.*

The notation above mentioned, as well as that adopted throughout the present paper, has the advantage of expressing something concerning the nature of the determinant; but for a mere abbreviation the following is convenient,

$$|\ 1,2, \cdot\cdot\ n\ |$$

and will be occasionally used.

The same property which gave rise to Theorem I. also leads without difficulty to that expressed by the following equation,

$$\begin{vmatrix} (1,1)+(1,1)' & (1,2)\cdot\cdot(1,n) \\ (2,1)+(2,1)' & (2,2)\cdot\cdot(2,n) \\ \cdot & \cdot\ \cdot\cdot\ \cdot \\ (n,1)+(n,1)' & (n,2)\cdot\cdot(n,n) \end{vmatrix} = \begin{vmatrix} (1,1) & (1,2)\cdot\cdot(1,n) \\ (2,1) & (2,2)\cdot\cdot(2,n) \\ \cdot & \cdot\ \cdot\cdot\ \cdot \\ (n,1) & (n,2)\cdot\cdot(n,n) \end{vmatrix} + \begin{vmatrix} (1,1)' & (1,2)\cdot\cdot(1,n) \\ (2,1)' & (2,2)\cdot\cdot(2,n) \\ \cdot & \cdot\ \cdot\cdot\ \cdot \\ (n,1)' & (n,2)\cdot\cdot(n,n) \end{vmatrix}$$

A more general form of which is easily seen to be also true, thus,

$$\begin{vmatrix} (1,1) + (1,1)' + \cdot\cdot & (1,2) + (1,2)' + \cdots\cdot & (1,n) + (1,n)' + \cdot\cdot \\ (2,1) + (2,1)' + \cdot\cdot & (2,2) + (2,2)' + \cdots\cdot & (2,n) + (2,n)' + \cdot\cdot \\ \cdot & \cdot \quad \cdot\cdot & \cdot \\ (n,1) + (n,1)' + \cdot\cdot & (n,2) + (n,2)' + \cdots\cdot & (n,n) + (n,n)' + \cdot\cdot \end{vmatrix}$$

$$= \begin{vmatrix} (1,1) & (1,2) \cdot\cdot & (1,n) \\ (2,1) & (2,2) \cdot\cdot & (2,n) \\ \cdot & \cdot \quad \cdot\cdot & \cdot \\ (n,1) & (n,2) \cdot\cdot & (n,n) \end{vmatrix} + \begin{vmatrix} (1,1)' & (1,2) \cdot\cdot & (1,n) \\ (2,1)' & (2,2) \cdot\cdot & (2,n) \\ \cdot & \cdot \quad \cdot\cdot & \cdot \\ (n,1)' & (n,2) \cdot\cdot & (n,n) \end{vmatrix} + \cdot\cdot + \begin{vmatrix} (1,1)' & (1,2)' \cdot\cdot & (1,n)' \\ (2,1)' & (2,2)' \cdot\cdot & (2,n)' \\ \cdot & \cdot \quad \cdot\cdot & \cdot \\ (n,1)' & (n,2)' \cdot\cdot & (n,n)' \end{vmatrix} + \cdot\cdot$$

Hence the following theorem may be enunciated:

> THEOREM II. *The determinant each of whose constituents is the sum of several others is equal to the sum of the determinants formed by all possible combinations of vertical rows, one being taken out of each pair found in the given determinant.*

If the number of terms in the first vertical row be p, that in the second q, and so on, the number of determinants will be $p\,q\,\ldots$

It may further be observed that if any vertical row of constituents, such as $(1,1)'$, $(2,1)'$,.., be identical with any other, such as $(1,2)'$, $(2,2)'$,.., the determinant containing those rows will vanish.

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§ III.—*Transformation of Determinants.*

IT was seen in the preceding section that a determinant is the sum of all possible combinations formed by taking one, and only one, constituent from each horizontal and each vertical row (the proper signs being affixed to these products); and it therefore follows that the absolute value of the determinant will not be altered by an interchange of all the first and second members of each of its constituents, i.e. by writing (q,p) for (p,q) throughout; nor will the sign be changed, for if it were, the sign of each term would be so also; but in the transformed expression the term

$$(1,1)(2,2)\cdot\cdot(n,n)$$

retains the same sign as before; so that the transformed determinant will be equivalent to the original one; i.e.

$$\begin{vmatrix} (1,1)(1,2)\cdot\cdot(1,n) \\ (2,1)(2,2)\cdot\cdot(2,n) \\ \cdot\quad\cdot\quad\cdot\cdot\quad\cdot \\ (n,1)(n,2)\cdot\cdot(n,n) \end{vmatrix} = \begin{vmatrix} (1,1)(2,1)\cdot\cdot(n,1) \\ (1,2)(2,2)\cdot\cdot(n,2) \\ \cdot\quad\cdot\quad\cdot\cdot\quad\cdot \\ (1,n)(2,n)\cdot\cdot(n,n) \end{vmatrix}$$

Hence,

THEOREM III. *The value of a determinant is not altered if the horizontal rows are written vertically, and the vertical horizontally.*

Again, it is seen from the expressions developed in § II. that in determinants of the orders 1, 2, . ., if two vertical (or horizontal) rows be interchanged the sign of the determinant itself is changed; and if this hold good for determinants of the $(n-1)$th order, the signs of all the determinants on the right-hand side of the expression for ∇, with the exception of the two consecutives, each of which contains only one of the rows in question, will always be changed.

The pair of terms (before the interchange of rows) is

$$\begin{array}{c} (1,i-1) \\ \\ (\alpha) \\ \\ \end{array}\begin{vmatrix} (1,i)(1,i+1)\cdot\cdot(1,i-2) \\ (2,i)(2,i+1)\cdot\cdot(2,i-2) \\ \cdot\quad\cdot\quad\cdot\cdot\quad\cdot \\ (n,i)(n,i+1)\cdot\cdot(n,i-2) \end{vmatrix} \qquad \begin{array}{c} (1,i) \\ \\ (\beta) \\ \\ \end{array}\begin{vmatrix} (1,i+1)(1,i+2)\cdot\cdot(1,i-1) \\ (2,i+1)(2,i+2)\cdot\cdot(2,i-1) \\ \cdot\quad\cdot\quad\cdot\cdot\quad\cdot \\ (n,i+1)(n,i+2)\cdot\cdot(n,i-1) \end{vmatrix}$$

and have the same or different signs, according as n is odd or even. They become by the interchange

$$\begin{array}{c} (1,i) \\ \\ (\alpha') \\ \\ \end{array}\begin{vmatrix} (1,i-1)(1,i+1)\cdot\cdot(1,i-2) \\ (2,i-1)(2,i+1)\cdot\cdot(1,i-2) \\ \cdot\quad\cdot\quad\cdot\cdot\quad\cdot \\ (n,i-1)(n,i+1)\cdot\cdot(n,i-2) \end{vmatrix} \qquad \begin{array}{c} (1,i-1) \\ \\ (\beta') \\ \\ \end{array}\begin{vmatrix} (1,i+1)(1,i+2)\cdot\ (1,i) \\ (2,i+1)(2,i+2)\cdot\cdot(2,i) \\ \cdot\quad\cdot\quad\cdot\cdot\quad\cdot \\ (n,i+1)(n,i+2)\cdot\cdot(n,i) \end{vmatrix}$$

Now if n be odd, these expressions have in their present forms the same explicit signs (both +) as (α) and (β); but when we remove the first vertical row in each determinant to the end, their sign is changed. If n be even, the removal of the first row to the end does not change the sign, but then (α') and (β') acquire opposite signs to (α) and (β) by the first interchange of rows.

So that whether n be odd or even the result will be a change of sign, and consequently the sign of the whole determinant will be changed;

and since this is the case for any two consecutive rows, it will be the case for any other pair of rows, since one of the pair will have to make an odd number and the other an even number of interchanges; hence finally,

$$\begin{vmatrix} (1,1)\,(1,2)\cdot\cdot(1,i)\cdot\cdot(1,j)\cdot\cdot(1,n) \\ (2,1)\,(2,2)\cdot\cdot(2,i)\cdot\cdot(2,j)\cdot\cdot(2,n) \\ \cdot\cdot\quad\cdot\quad\cdot\cdot\quad\cdot\quad\cdot\cdot\quad\cdot\quad\cdot\cdot\quad\cdot \\ (n,1)\,(n,2)\cdot\cdot(n,i)\cdot\cdot(n,j)\cdot\cdot(n,n) \end{vmatrix} = - \begin{vmatrix} (1,1)\,(1,2)\cdot\cdot(1,j)\cdot\cdot(1,i)\cdot\cdot(1,n) \\ (2,1)\,(2,2)\cdot\cdot(2,j)\cdot\cdot(2,i)\cdot\cdot(2,n) \\ \cdot\quad\cdot\quad\cdot\cdot\quad\cdot\quad\cdot\cdot\quad\cdot\quad\cdot\cdot\quad\cdot \\ (n,1)\,(n,2)\cdot\cdot(n,j)\cdot\cdot(n,i)\cdot\cdot(n,n) \end{vmatrix}$$

Hence,

THEOREM IV. *If two vertical or horizontal rows are interchanged the sign of the determinant is changed.*

and consequently, when two vertical (or horizontal) rows become identical, the determinant will be equal to itself with its sign changed, in other words it will vanish, i.e.

$$\begin{vmatrix} (1,1)\,(1,2)\cdot\cdot(1,i)\cdot\cdot(1,i)\cdot\cdot(1,n) \\ (2,1)\,(2,2)\cdot\cdot(2,i)\cdot\cdot(2,i)\cdot\cdot(2,n) \\ \cdot\quad\cdot\quad\cdot\cdot\quad\cdot\quad\cdot\cdot\quad\cdot\quad\cdot\quad\cdot \\ (n,1)\,(n,2)\cdot\cdot(n,i)\cdot\cdot(n,i)\cdot\cdot(n,n) \end{vmatrix} = 0$$

Hence,

THEOREM V. *If two vertical or horizontal rows become identical, the determinant vanishes.*

In certain cases a determinant degenerates into the product of two others; thus, if

$$\begin{array}{lll} (i,1) = 0, & (i,2) = 0, & \cdot\cdot\ (i,i-1) = 0 \\ (i+1,1) = 0, & (i+1,2) = 0, & \cdot\cdot\ (i+1,i-1) = 0 \\ \quad\cdot\qquad\cdot & \quad\cdot\qquad\cdot & \cdot\cdot\qquad\cdot\qquad\cdot \\ (n,1) = 0, & (n,2) = 0, & \cdot\cdot\ (n,i-1) = 0 \end{array}$$

then the determinant

$$= \begin{vmatrix} (1,1) & (1,2) & \cdot\cdot & (1,i) & \cdot\cdot & (1,n) \\ (2,1) & (2,2) & \cdot\cdot & (2,i) & \cdot\cdot & (2,n) \\ \cdot & \cdot & \cdot\cdot & \cdot & \cdot\cdot & \cdot \\ (i-1,1) & (i-1,2) & \cdot\cdot & (i-1,i) & \cdot\cdot & (i-1,n) \\ * & * & \cdot\cdot & (i,i) & \cdot\cdot & (i,n) \\ \cdot & \cdot & \cdot\cdot & \cdot & \cdot\cdot & \cdot \\ * & * & \cdot\cdot & (n,i) & \cdot\cdot & (n,n) \end{vmatrix},$$

may be successively reduced until the determinant

$$\begin{vmatrix} (i,i) & (i,i+1) & \cdot\cdot & (i,n) \\ (i+1,i) & (i+1,i+1) & \cdot\cdot & (i+1,n) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (n,i) & (n,i+1) & \cdot\cdot & (n,n) \end{vmatrix}$$

is a common factor of all the terms, the other determinants of the order $n-i$ vanishing, for at the $(i-1)$th stage of reduction there will remain only the determinants formed from the $(n-i+1)$ lower rows; but since there are only $(n-i+1)$ vertical rows in this group, there can be only one determinant, viz. that mentioned above. Again, taking the first vertical row as the primary, and developing, it would be found that the determinant may be reduced until

$$\begin{vmatrix} (1,1) & (1,2) & \cdot\cdot & (1,i-1) \\ (2,1) & (2,2) & \cdot\cdot & (2,i+1) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (i-1,1) & (i-1,2) & \cdot\cdot & (i-1,i-1) \end{vmatrix}$$

is a common factor; but as the whole determinant is homogeneous, and of the order n, it follows that its absolute value is equal to the product of these two last determinants; moreover the sign of the product will be the same as that of the given determinant, since the signs of the terms

$$\begin{array}{llll} (1,1) & (2,2) & \cdot\cdot & (n,n) \\ (1,1) & (2,2) & \cdot\cdot & (i-1,i-1) \\ (i,i) & (i+1,i+1) & \cdot\cdot & (n,n) \end{array}$$

are all +; hence

$$\begin{vmatrix} (1,1) & (1,2) & \cdot\cdot & (1,i) & \cdot\cdot & (1,n) \\ (2,1) & (2,2) & \cdot\cdot & (2,i) & \cdot\cdot & (2,n) \\ \cdot & \cdot & \cdot\cdot & \cdot & \cdot\cdot & \cdot \\ (i-1,1) & (i-1,2) & \cdot\cdot & (i-1,i) & \cdot\cdot & (1-1,n) \\ \ast & \ast & \cdot\cdot & (i,i) & \cdot\cdot & (i,n) \\ \cdot & \cdot & \cdot\cdot & \cdot & \cdot\cdot & \\ \ast & \ast & \cdot\cdot & (n,i) & \cdot\cdot & (n,n) \end{vmatrix}.$$

$$= \begin{vmatrix} (1,1) & (1,2) & \cdot\cdot & (1,i-1) \\ (2,1) & (2,2) & \cdot\cdot & (2,i-1) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (i-1,1) & (i-1,2) & \cdot\cdot & (i-1,i-1) \end{vmatrix} . \begin{vmatrix} (i,i) & (i,i+1) & \cdot\cdot & (i,n) \\ (i+1,i) & (i+1,i+1) & \cdot\cdot & (i+1,n) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (n,i) & (n,i+1) & \cdot\cdot & (n,n) \end{vmatrix}$$

and similarly, if another set of constituents vanished, one of these latter determinants would be equal to the product of two others, and the whole determinant would be equal to the product of three determinants, and so on.

Hence,

THEOREM VI. *If in the last* (n – i) *horizontal rows of a determinant all, excepting the last* (n – i) *vertical rows, vanish, the determinant will be equal to the product of the determinants formed respectively from the first* i *and the last* (n – i) *horizontal and vertical rows.*

This theorem involves also the following:

THEOREM VII. *If in one of the parallelograms which is a complement of two squares about the diagonal of a determinant all the constituents vanish, then all those in the other complement may be put* = 0, *without altering the value of the determinant.*

Amongst other particular cases the following may be noticed,

$$\begin{vmatrix} 1 & \ast & \cdot\cdot & \ast & \ast & \cdot\cdot & \ast \\ \ast & 1 & \cdot\cdot & \ast & \ast & \cdot\cdot & \ast \\ \cdot & \cdot & \cdot\cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ \ast & \ast & \cdot\cdot & (1,1) & (1,2) & \cdot\cdot & (1,n) \\ \ast & \ast & \cdot\cdot & (2,1) & (2,2) & \cdot\cdot & (2,n) \\ \cdot & \cdot & \cdot\cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ \ast & \ast & \cdot\cdot & (n,1) & (n,2) & \cdot\cdot & (n,n) \end{vmatrix} = \begin{vmatrix} (1,1) & (1,2) & \cdot\cdot & (1,n) \\ (2,1) & (2,2) & \cdot\cdot & (2,n) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (n,1) & (n,2) & \cdot\cdot & (n,n) \end{vmatrix}$$

$$\begin{vmatrix} (1,1) & (1,2) & \cdot\cdot & (1,n) \\ \ast & (2,2) & \cdot\cdot & (2,n) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ \ast & \ast & \cdot\cdot & (n,n) \end{vmatrix} = (1,1)\ .\ (2,2)\ \ldots\ (n,n)$$

which will suggest many others.

The principles of the present and preceding sections enable us also to establish another theorem; consider the determinant $| 1,2, \ldots n |$, and let $\alpha, \beta, \ldots \nu, \kappa$ be any whole numbers such that

$$\alpha+\beta+\cdot\cdot+\nu+\kappa=n$$

then by taking α vertical rows out of the first α horizontal rows, and forming the determinant

$$\begin{vmatrix} (1,1) & (1,2) & \cdot\cdot & (1,\alpha) \\ (2,1) & (2,2) & \cdot\cdot & (2,\alpha) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (\alpha,1) & (\alpha,2) & \cdot\cdot & (\alpha,\alpha) \end{vmatrix} = | 1,2, \cdot\cdot \alpha |$$

and, similarly, taking the next β vertical rows out of the next β horizontal rows, and forming the determinant

$$\begin{vmatrix} (\alpha+1,\alpha+1) & (\alpha+1,\alpha+2) & \cdot\cdot & (\alpha+1,\beta) \\ (\alpha+2,\alpha+1) & (\alpha+2,\alpha+2) & \cdot\cdot & (\alpha+2,\beta) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (\beta,\alpha+1) & (\beta,\alpha+2) & \cdot\cdot & (\beta,\beta) \end{vmatrix} = |\,\alpha+1,\alpha+2,\,\cdot\cdot\,\beta\,|$$

and so on, until last κ vertical rows be taken out of the last κ horizontal rows; then forming the determinant

$$\begin{vmatrix} (\nu+1,\nu+1) & (\nu+1,\nu+2) & \cdot\cdot & (\nu+1,\kappa) \\ (\nu+2,\nu+1) & (\nu+2,\nu+2) & \cdot\cdot & (\nu+2,\kappa) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (\kappa,\nu+1) & (\kappa,\nu+2) & \cdot\cdot & (\kappa,\kappa) \end{vmatrix} = |\,\nu+1,\nu+2,\,\cdot\cdot\,\kappa\,|$$

it is clear that the product of the determinants will give all the terms arising from the constituents which lie on the squares about the diagonal of $|\,1,2,\,..\,n\,|$; and by extending the sign of summation to all combinations in which no vertical row is twice employed, all the combinations in the determinant $|\,1,2,\,..\,n\,|$ will be produced; and if moreover the sign of the product be made positive or negative, according as it requires an even or odd number of interchanges of vertical rows in $|\,1,2,\,..\,n\,|$ to bring the determinants so formed all upon the diagonal of $|\,1,2,\,..\,n\,|$, there will result,

$$|1,2,\cdot\cdot\, n| = \Sigma\{\pm\,|1,2,\cdot\cdot\,\alpha|\,|\alpha+1,\alpha+2,\cdot\cdot\,\beta|\;\cdot\cdot\;|\nu+1,\nu+2,\cdot\cdot\,\kappa|\}$$

Hence the following theorem:

THEOREM VIII. *If in a determinant of the* n*th order* $\alpha, \beta, \ldots \kappa$ *be whole numbers such that* $\alpha+\beta\,..+\kappa=n$, *the determinant may be expressed as the sum of the products of the determinants formed from all the groups of* α *vertical rows in the first* α *horizontal rows, from all the groups of* β *vertical rows in the next* β *horizontal rows, and so on, it being observed that no vertical row is to be twice employed.*

Thus for example;

$$\begin{vmatrix} (1,1) & (1,2) & (1,3) & (1,4) \\ (2,1) & (2,2) & (2,3) & (2,4) \\ (3,1) & (3,2) & (3,3) & (3,4) \\ (4,1) & (4,2) & (4,3) & (4,4) \end{vmatrix}$$

$$= \begin{vmatrix} (1,1)(1,2) \\ (2,1)(2,2) \end{vmatrix} \begin{vmatrix} (3,3)(3,4) \\ (4,3)(4,4) \end{vmatrix} + \begin{vmatrix} (1,3)(1,4) \\ (2,3)(2,4) \end{vmatrix} \begin{vmatrix} (3,1)(3,2) \\ (4,1)(4,2) \end{vmatrix} + \begin{vmatrix} (1,1)(1,4) \\ (2,1)(2,4) \end{vmatrix} \begin{vmatrix} (3,2)(3,3) \\ (4,2)(4,3) \end{vmatrix}$$

$$- \begin{vmatrix} (1,1)(1,3) \\ (2,1)(2,3) \end{vmatrix} \begin{vmatrix} (3,2)(3,4) \\ (4,2)(4,4) \end{vmatrix} - \begin{vmatrix} (1,2)(1,3) \\ (2,2)(2,3) \end{vmatrix} \begin{vmatrix} (3,4)(3,1) \\ (4,4)(4,1) \end{vmatrix} - \begin{vmatrix} (1,2)(1,4) \\ (2,2)(2,4) \end{vmatrix} \begin{vmatrix} (3,1)(3,3) \\ (4,1)(4,3) \end{vmatrix}$$

The following are examples of the application of these theorems:

Let there be four planes intersecting in a point, two of them passing through the axis of z, their equations will then be

$$\begin{aligned} lx + my + nz + k &= 0 \\ l_1x + m_1y + n_1z + k_1 &= 0 \\ l_2x + m_2y &= 0 \\ l_3x + m_3y &= 0 \end{aligned}$$

and the determinant formed from them will degenerate into the product of two others, thus,

$$\begin{vmatrix} l_2 & m_2 \\ l_3 & m_3 \end{vmatrix} \begin{vmatrix} n & k \\ n_1 & k_1 \end{vmatrix} = 0$$

which is satisfied by either of the following equations,

$$\begin{vmatrix} l_2 & m_2 \\ l_3 & m_3 \end{vmatrix} = 0, \quad \begin{vmatrix} n & k \\ n_1 & k_1 \end{vmatrix} = 0$$

the first of which is the condition that the third and fourth planes shall coincide; the second expresses that the four planes intersect in the axis of z at a point where

$$-z = \frac{k}{n} = \frac{k_1}{n_1}$$

Hence, either the four planes intersect in this point, or the third and fourth coincide.

As another example; the equations to a cone and its reciprocal have been shown to be

$$Ax^2 + By^2 + Cz^2 + 2(Fyz + Gzx + Hxy) = 0$$

$$\begin{vmatrix} A & H & G & \xi \\ H & B & F & \eta \\ G & F & C & \zeta \\ \xi & \eta & \zeta & 0 \end{vmatrix} = 0$$

and if

$$A = 0, \ B = 0, \ H = 0$$

the first degenerates into the two planes,

$$z = 0, \ 2Gx + 2Fy + Cz = 0$$

and the second into the two coincident planes,

$$\begin{vmatrix} G & \xi \\ F & \eta \end{vmatrix} \begin{vmatrix} G & F \\ \xi & \eta \end{vmatrix} = \begin{vmatrix} G & \xi \\ F & \eta \end{vmatrix}^2 = 0$$

which are perpendicular to both of the former planes, as they should be.

§ IV.—*On the Connexion between Determinants and Linear Equations.*

It was seen above that determinants of the order 1, 2, . . are the results of the elimination of 1, 2, . . variables from the same number of linear equations; suppose that this holds good for n variables, then

$$\begin{vmatrix} (1,1) & (1,2) & \cdot\cdot & (1,n) \\ (2,1) & (2,2) & \cdot\cdot & (2,n) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (n,1) & (n,2) & \cdot\cdot & (n,n) \end{vmatrix} = 0$$

will be the result of the elimination of $x_1, x_2, \ldots x_n$ from the equations

$$\begin{gathered} (1,1)x_1 + (1,2)x_2 + \cdot\cdot + (1,n)x_n = 0 \\ (2,1)x_1 + (2,2)x_2 + \cdot\cdot + (2,n)x_n = 0 \\ \cdot \quad\quad \cdot \quad\quad \cdot\cdot \quad\quad \cdot \\ (n,1)x_1 + (n,2)x_2 + \cdot\cdot + (n,n)x_n = 0 \end{gathered}$$

If, however, the second members of these equations instead of being zero are $u_1, u_2, \ldots u_n$, the system may be written thus,

$$\begin{gathered} \left((1,1) - \frac{u_1}{x_1}\right) x_1 + (1,2)x_2 + \cdot\cdot + (1,n)x_n = 0 \\ \left((2,1) - \frac{u_2}{x_1}\right) x_1 + (2,2)x_2 + \cdot\cdot + (2,n)x_n = 0 \\ \cdot \quad\quad \cdot \quad\quad \cdot\cdot \quad\quad \cdot \\ \left((n,1) - \frac{u_n}{x_1}\right) x_1 + (n,2)x_2 + \cdot\cdot + (n,n)x_n = 0 \end{gathered}$$

and the following determinant deduced,

$$\begin{vmatrix} (1,1) - \frac{u_1}{x_1} & (1,2) & \cdot\cdot & (1,n) \\ (2,1) - \frac{u_2}{x_1} & (2,2) & \cdot\cdot & (2,n) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (n,1) - \frac{u_n}{x_1} & (n,2) & \cdot\cdot & (n,n) \end{vmatrix} = 0$$

or by Theorem I.

$$\begin{vmatrix} (1,1) & (1,2) & \cdot\cdot & (1,n) \\ (2,1) & (2,2) & \cdot\cdot & (2,n) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (n,1) & (n,2) & \cdot\cdot & (n,n) \end{vmatrix} x_1 = \begin{vmatrix} u_1 & (1,2) & \cdot\cdot & (1,n) \\ u_2 & (2,2) & \cdot\cdot & (2,n) \\ & \cdot & \cdot\cdot & \cdot \\ u_n & (n,2) & \cdot\cdot & (n,n) \end{vmatrix}$$

with similar expressions for $x_2, x_3, \ldots x_n$; so that the given equations are completely solved. If moreover

$$u_1 = -(1,0)x, \quad u_2 = -(2,0)x \cdots u_n = -(n,0)x$$

it would be found, by the method employed above, and by Theorem IV., that

$$x : x_1 : \cdots x_n$$

$$= \begin{vmatrix} (1,1) & (1,2) & \cdots & (1,n) \\ (2,1) & (2,2) & \cdots & (2,n) \\ \cdot & \cdot & \cdots & \cdot \\ (n,1) & (n,2) & \cdots & (n,n) \end{vmatrix} : \pm \begin{vmatrix} (1,2) & (1,3) & \cdots & (1,0) \\ (2,2) & (2,3) & \cdots & (2,0) \\ \cdot & \cdot & \cdots & \cdot \\ (n,2) & (n,3) & \cdots & (n,0) \end{vmatrix} : \cdots \pm \begin{vmatrix} (1,2) & (1,1) & \cdots & (1,n-1) \\ (2,0) & (2,1) & \cdots & (2,n-1) \\ \cdot & \cdot & \cdots & \cdot \\ (n,0) & (n,1) & \cdots & (n,n-1) \end{vmatrix}$$

the upper or lower signs being taken according as $(n+1)$ is odd or even. And if in addition to the given equations there exist the relation

$$(0,0)x + (0,1)x_1 + \cdots + (0,n)x_n = 0$$

the substitution of the values of the ratios $x : x_1 : \ldots x_n$ from the previous equations will give rise to the determinant

$$\begin{vmatrix} (0,0) & (0,1) & \cdots & (0,n) \\ (1,0) & (1,1) & \cdots & (1,n) \\ \cdot & \cdot & \cdots & \cdot \\ (n,0) & (n,1) & \cdots & (n,n) \end{vmatrix} = 0$$

which is therefore the result of the elimination of $x, x_1, \ldots x_n$ from the $(n+1)$ linear equations

$$\begin{aligned} (0,0)x + (0,1)x_1 + \cdots + (0,n)x_n &= 0 \\ (1,0)x + (1,1)x_1 + \cdots + (1,n)x_n &= 0 \\ \cdot \qquad \cdot \qquad \cdots \qquad \cdot \quad & \\ (n,0)x + (n,1)x_1 + \cdots + (n,n)x_n &= 0 \end{aligned}$$

And as it has been shown that this holds good in the cases where $n = 1$, $n = 2$, . ., it follows,

Theorem IX. *A determinant of the order* n *is in general the result of the elimination of* n *variables from* n *linear equations, whose coefficients are the constituents of the determinant.*

Conversely,

Theorem X. *If a determinant of the* n*th order vanishes, a system of* n *homogeneous linear equations, the coefficients of which are the constituents of the given determinant, may always be established.*

By Theorem III. it also appears that this theorem holds good whether the determinant be resolved according to its vertical or its horizontal rows.

This being the case, recourse will be had to the properties of linear equations whenever they will simplify the establishment of theorems relating to determinants.

Besides the cases noticed in the introductory section, the following are examples of this theorem.

The condition that three straight lines may be parallel to one plane will be given by the elimination of x, y, z, from the equations,

$$\begin{aligned} lx + my + nz &= 0 \\ l_1x + m_1y + n_1z &= 0 \\ l_2x + m_2y + n_2z &= 0 \end{aligned}$$

i.e. by the determinant

$$\begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

The condition that four planes may pass through a point will be given by the elimination of x, y, z, from the equations

$$\begin{aligned} lx + my + nz + k &= 0 \\ l_1x + m_1y + n_1z + k_1 &= 0 \\ l_2x + m_2y + n_2z + k_2 &= 0 \\ l_3x + m_3y + n_3z + k_3 &= 0 \end{aligned}$$

i.e. by the determinant

$$\begin{vmatrix} l & m & n & k \\ l_1 & m_1 & n_1 & k_1 \\ l_2 & m_2 & n_2 & k_2 \\ l_3 & m_3 & n_3 & k_3 \end{vmatrix} = 0$$

all of which when developed will be found to agree with the usual conditions.

The following example is of frequent occurrence in geometrical questions. To find the equation to the cone reciprocal to the cone

$$\mathrm{A}x^2 + \mathrm{B}y^2 + \mathrm{C}z^2 + 2\,(\mathrm{F}yz + \mathrm{G}zx + \mathrm{H}xy) = 0$$

If ξ, η, ζ be the co-ordinates of a point on the reciprocal cone, the conditions that the radius vector of this point shall be perpendicular to the tangent plane along the line containing the point (x, y, z) will be

$$\begin{aligned} \mathrm{A}x + \mathrm{H}y + \mathrm{G}z + \theta\xi &= 0 \\ \mathrm{H}x + \mathrm{B}y + \mathrm{F}z + \theta\eta &= 0 \\ \mathrm{G}x + \mathrm{F}y + \mathrm{C}z + \theta\zeta &= 0 \\ \xi x + \eta y + \zeta z \qquad &= 0 \end{aligned}$$

Hence eliminating x, y, z, θ, together, the equation to the reciprocal cone will be

$$\begin{vmatrix} A & H & G & \xi \\ H & B & F & \eta \\ G & F & C & \zeta \\ \xi & \eta & \zeta & 0 \end{vmatrix} = 0$$

The equation to the given cone may also be thrown into the form of a determinant; for writing the above equations in the following manner,

$$\begin{aligned} &\theta\xi + 0 + 0 + Ax + Hy + Gz = 0 \\ &0 + \theta\eta + 0 + Hx + By + Fz = 0 \\ &0 + 0 + \theta\zeta + Gx + Fy + Cz = 0 \\ &\theta(x\xi + y\eta + z\zeta) \quad + 0 \quad = 0 \end{aligned}$$

and eliminating $\theta\xi$, $\theta\eta$, $\theta\zeta$, there results

$$\begin{vmatrix} 1 & 0 & 0 & Ax + Hy + Gz \\ 0 & 1 & 0 & Hx + By + Fz \\ 0 & 0 & 1 & Gx + Fy + Cz \\ x & y & z & 0 \end{vmatrix} = 0$$

the same method is obviously applicable to any surface of the second order, and the equation

$$Ax^2 + By^2 + Cz^2 + 2(Fyz + Gzx + Hxy) + Lx + My + Nz - K = 0$$

may be written in either of the following ways,

$$\begin{vmatrix} 1 & 0 & 0 & Ax + Hy + Cz + L \\ 0 & 1 & 0 & Hx + By + Fz + M \\ 0 & 0 & 1 & Gx + Fy + Cz + N \\ x & y & z & K \end{vmatrix} = 0$$

or

$$\begin{vmatrix} 1 & 0 & 0 & 0 & Ax + Hy + Gz \\ 0 & 1 & 0 & 0 & Hx + By + Fz \\ 0 & 0 & 1 & 0 & Gx + Fy + Cz \\ 0 & 0 & 0 & 1 & Lx + My + Nz \\ x & y & z & 1 & K \end{vmatrix} = 0$$

Another form of the equation to a surface of the second order, similar to that to the reciprocal cone, will be given hereafter.

§ V.—*On the Products and Powers of Determinants.*

Consider the same system of linear equations as before, and also the derived system,

$$\begin{array}{l}
(1,1)'u_1 + (1,2)'u_2 + \cdot\cdot + (1,n)'u_n = v_1 \\
(2,1)'u_1 + (2,2)'u_2 + \cdot\cdot + (2,n)'u_n = v_2 \\
\quad . \qquad\quad . \qquad \cdot\cdot \qquad . \\
(n,1)'u_1 + (n,2)'u_2 + \cdot\cdot + (n,n)'u_n = v_n
\end{array}$$

or

$$\begin{array}{l}
\{(1,1)'(1,1) + (1,2)'(2,1) + \cdot\cdot\}x_1 + \{(1,1)'(1,2) + (1,2)'(2,2) + \cdot\cdot\}x_2 + \cdot\cdot = v_1 \\
\{(2,1)'(1,1) + (2,2)'(2,1) + \cdot\cdot\}x_1 + \{(2,1)'(1,2) + (2,2)'(2,2) + \cdot\cdot\}x_2 + \cdot\cdot = v_2 \\
\quad . \qquad\qquad\qquad . \qquad\qquad\qquad\qquad . \qquad\qquad . \qquad \cdot\cdot \qquad . \\
\{(n,1)'(1,1) + (n,2)'(2,1) + \cdot\cdot\}x_1 + \{(n,1)'(1,2) + (n,2)'(2,2) + \cdot\cdot\}x_2 + \cdot\cdot = v_n
\end{array}$$

the latter system then gives

$$\left|\begin{array}{cccc}
(1,1)'(1,1)+(1,2)'(2,1)+\cdot\cdot & (1,1)'(1,2)+(1,2)'(2,2)+\cdot\cdot & \cdot\cdot & (1,1)'(1,n)+(1,2)'(2,n)+\cdot\cdot \\
(2,1)'(1,1)+(2,2)'(2,1)+\cdot\cdot & (2,1)'(1,2)+(2,2)'(2,2)+\cdot\cdot & \cdot\cdot & (2,1)'(1,n)+(2,2)'(2,n)+\cdot\cdot \\
. & . & \cdot\cdot & . \\
(n,1)'(1,1)+(n,2)'(2,1)+\cdot\cdot & (n,1)'(1,2)+(n,2)'(2,2)+\cdot\cdot & \cdot\cdot & (n,1)'(1,n)+(n,2)'(2,n)+\cdot\cdot
\end{array}\right| x_1$$

$$= \left|\begin{array}{cccc}
v_1 & v_2 & \cdot\cdot & v_n \\
(2,1)'(1,1)+(2,2)'(2,1)+\cdot\cdot & (2,1)'(1,2)+(2,2)'(2,2)+\cdot\cdot & \cdot\cdot & (2,1)'(1,n)+(2,2)'(2,n)+\cdot\cdot \\
. & . & \cdot\cdot & . \\
(n,1)'(1,1)+(n,2)'(2,1)+\cdot\cdot & (n,1)'(1,2)+(n,2)'(2,2)+\cdot\cdot & \cdot\cdot & (n,1)'(1,n)+(n,2)'(2,n)+\cdot\cdot
\end{array}\right|$$

on the other hand, writing the two systems of linear equations as one system thus,

$$\begin{array}{ccccccccccccccccc}
(1,1)x_1 & + & (1,2)x_2 & + \cdot\cdot + & (1,n)x_n & - & u_1 & + & * & + \cdot\cdot + & * & = 0 \\
(2,1)x_1 & + & (2,2)x_2 & + \cdot\cdot + & (2,n)x_n & + & * & - & u_2 & + \cdot\cdot + & * & = 0 \\
. & & . & \cdot\cdot & . & & . & & . & \cdot\cdot & . & . \\
(n,1)x_1 & + & (n,2)x_2 & + \cdot\cdot + & (n,n)x_n & + & * & + & * & + \cdot\cdot - & u_n & = 0 \\
* & + & * & + \cdot\cdot + & * & + & (1,1)'u_1 & + & (1,2)'u_2 & + \cdot\cdot + & (1,n)'u_n & = v_1 \\
* & + & * & + \cdot\cdot + & * & + & (2,1)'u_1 & + & (2,2)'u_2 & + \cdot\cdot + & (2,n)'u_n & = v_2 \\
. & & . & \cdot\cdot & . & & . & & . & \cdot\cdot & . & . \\
* & + & * & + \cdot\cdot + & * & + & (n,1)'u_1 & + & (n,2)'u_2 & + \cdot\cdot + & (n,n)'u_n & = v_n
\end{array}$$

there may be deduced,

$$\left|\begin{array}{cccccccc}
(1,1) & (2,1) & \cdot\cdot & (n,1) & \ast & \ast & \cdot\cdot & \ast \\
(1,2) & (2,2) & \cdot\cdot & (n,2) & \ast & \ast & \cdot\cdot & \ast \\
\cdot & \cdot & \cdot\cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\
(1,n) & (2,n) & \cdot\cdot & (n,n) & \ast & \ast & \cdot\cdot & \ast \\
-1 & \ast & \cdot\cdot & \ast & (1,1)' & (2,1)' & \cdot\cdot & (n,1)' \\
\ast & -1 & \cdot\cdot & \ast & (1,2)' & (2,2)' & \cdot\cdot & (n,2)' \\
\cdot & \cdot & \cdot\cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\
\ast & \ast & \cdot\cdot & -1 & (1,n)' & (2,n)' & \cdot\cdot & (n,n)'
\end{array}\right| x_1 = \left|\begin{array}{cccccccc}
\ast & \ast & \cdot\cdot & \ast & v_1 & v_2 & \cdot\cdot & v_n \\
(1,2) & (2,2) & \cdot\cdot & (n,2) & \ast & \ast & \cdot\cdot & \ast \\
\cdot & \cdot & \cdot\cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\
(1,n) & (2,n) & \cdot\cdot & (n,n) & \ast & \ast & \cdot\cdot & \ast \\
-1 & \ast & \cdot\cdot & \ast & (1,1)' & (2,1)' & \cdot\cdot & (n,1)' \\
\ast & -1 & \cdot\cdot & \ast & (1,2)' & (2,2)' & \cdot\cdot & (n,2)' \\
\cdot & \cdot & \cdot\cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\
\ast & \ast & \cdot\cdot & -1 & (1,n)' & (2,n)' & \cdot\cdot & (n,n)'
\end{array}\right|$$

so that comparing the two systems,

$$\left|\begin{array}{ccc}
(1,1)'(1,1)+(1,2)'(2,1)+\cdot\cdot & (1,1)'(1,2)+(1,2)'(2,2)+\cdot\cdot & \cdot\cdot(1,1)'(1,n)+(1,2)'(2,n)+\cdot\cdot \\
(2,1)'(1,1)+(2,2)'(2,1)+\cdot\cdot & (2,1)'(1,2)+(2,2)'(2,2)+\cdot\cdot & \cdot\cdot(2,1)'(1,n)+(2,2)'(2,n)+\cdot\cdot \\
\cdot \quad \cdot & \cdot \quad \cdot & \cdot\cdot \quad \cdot \\
(n,1)'(1,1)+(n,2)'(2,1)+\cdot\cdot & (n,1)'(1,2)+(n,2)'(2,2)+\cdot\cdot & \cdot\cdot(n,1)'(1,n)+(n,2)'(2,n)+\cdot\cdot
\end{array}\right|$$

$$= \left|\begin{array}{cccc}
(1,1) & (1,2) & \cdot\cdot & (1,n) \\
(2,1) & (2,2) & \cdot\cdot & (2,n) \\
\cdot & \cdot & \cdot\cdot & \cdot \\
(n,1) & (n,2) & \cdot\cdot & (n,n)
\end{array}\right| \left|\begin{array}{cccc}
(1,1)' & (1,2)' & \cdot\cdot & (1,n)' \\
(2,1)' & (2,2)' & \cdot\cdot & (2,n)' \\
\cdot & \cdot & \cdot\cdot & \cdot \\
(n,1)' & (n,2)' & \cdot\cdot & (n,n)'
\end{array}\right|$$

With respect to the formation of the constituents of the product, it may be noticed that if the first member of the above equation be written thus,

$$\left|\begin{array}{cccc}
(1,1)'' & (1,2)'' & \cdot\cdot & (1,n)'' \\
(2,1)'' & (2,2)'' & \cdot\cdot & (2,n)'' \\
\cdot & \cdot & \cdot\cdot & \cdot \\
(n,1)'' & (n,2)'' & \cdot\cdot & (n,n)''
\end{array}\right|.$$

and the equation thus,

$$\nabla'' = \nabla\nabla'$$

then $(i,j)''$ is the sum of the products formed by taking in order the terms along the ith horizontal row of ∇' and multiplying them by the terms along the jth vertical row of ∇ (or vice versâ); or since the vertical rows may be changed into horizontal, it may be said that $(i,j)''$ is the sum of the products made by taking the terms along the ith horizontal row of ∇ with those along the jth horizontal row of ∇' (or vice versâ).

In the case where

$$\begin{array}{llll}
(1,1)' = (1,1) & (1,2)' = (2,1) & \cdots & (1,n)' = (n,1) \\
(2,1)' = (1,2) & (2,2)' = (2,2) & \cdots & (2,n)' = (n,2) \\
\cdot \quad \cdot & \cdot \quad \cdot & \cdot\cdot & \cdot \quad \cdot \\
(n,1)' = (1,n) & (n,2)' = (2,n) & \cdots & (n,n)' = (n,n)
\end{array}$$

the above expression becomes

$$\left|\begin{array}{cccc}
(1,1)^2 + (2,1)^2 + \cdots & (1,1)(1,2) + (2,1)(2,2) + \cdots & \cdots & (1,1)(1,n) + (2,1)(2,n) + \cdots \\
(1,2)(1,1) + (2,2)(2,1) + \cdots & (1,2)^2 + (2,2)^2 + \cdots & \cdots & (1,2)(1,n) + (2,2)(2,n) + \cdots \\
\cdot & \cdot & & \cdot \quad \cdot\cdot \\
(1,n)(1,1) + (2,n)(2,1) + \cdots & (1,n)(1,2) + (2,n)(2,2) + \cdots & \cdots & (1,n)^2 + (2,n)^2 + \cdots
\end{array}\right|$$

$$= \left|\begin{array}{cccc}
(1,1) & (1,2) & \cdots & (1,n) \\
(2,1) & (2,2) & \cdots & (2,n) \\
\cdot & \cdot & \cdot\cdot & \cdot \\
(n,1) & (n,2) & \cdots & (n,n)
\end{array}\right|^2$$

Conversely the product of two determinants or the square of a determinant may be resolved into a single determinant, as above.

Hence the following:

THEOREM XI. *A determinant whose constituents are linear functions of given constituents, the coefficients being the same for each horizontal row, is equal to the product of the two determinants whose constituents are the given constituents and the coefficients respectively.*

This formula is capable of many applications, some of which are shown in the following examples.

The condition that three straight lines lie in a plane is independent of the direction of the co-ordinate axes; in fact, the condition in question is, as was shown above,

$$\left|\begin{array}{ccc}
l & m & n \\
l' & m' & n' \\
l'' & m'' & n''
\end{array}\right| = 0$$

and if the directions of the co-ordinate axes be changed, this becomes

$$\left|\begin{array}{ccc}
\alpha l + \beta m + \gamma n & \alpha' l + \beta' m + \gamma' n & \alpha'' l + \beta'' m + \gamma'' n \\
\alpha l' + \beta m' + \gamma n' & \alpha' l' + \beta' m' + \gamma' n' & \alpha'' l' + \beta'' m' + \gamma'' n' \\
\alpha l'' + \beta m'' + \gamma n'' & \alpha' l'' + \beta' m'' + \gamma' n'' & \alpha'' l'' + \beta'' m'' + \gamma'' n''
\end{array}\right|$$

$$= \left|\begin{array}{ccc}
\alpha & \beta & \gamma \\
\alpha' & \beta' & \gamma' \\
\alpha'' & \beta'' & \gamma''
\end{array}\right| \left|\begin{array}{ccc}
l & m & n \\
l' & m' & n' \\
l'' & m'' & n''
\end{array}\right| = 0$$

which coincides with the original condition, since the first factor cannot vanish, as the three co-ordinate axes cannot be in one plane.

The condition that a surface of the second order has no centre is independent of the direction of the co-ordinate axes; in fact, the condition is

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = 0$$

which by a change of the direction of co-ordinate axes, and by writing for brevity

$$Ax^2 + \cdot\cdot = Ax^2 + By^2 + Cz^2 + 2\,(Fyz + Gzx + Hxy)$$

$$Ax\xi + \cdot\cdot = Ax\xi + By\eta + Cz\zeta + F(y\zeta + z\eta) + G(z\xi + x\zeta) + H(x\eta + y\xi)$$

$$\begin{vmatrix} Al^2 + \cdot\cdot & Alm + \cdot\cdot & Aln + \cdot\cdot \\ Aml + \cdot\cdot & Am^2 + \cdot\cdot & Amn + \cdot\cdot \\ Anl + \cdot\cdot & Anm + \cdot\cdot & An^2 + \cdot\cdot \end{vmatrix} = \begin{vmatrix} l & m & n \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix}^2 \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = 0$$

which, as in the former case, proves the proposition.

The following examples are taken from an interesting Mémoire by M. Joachimsthal. Crelle, tom. XXXIX.

Let the equation to a conic section be

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} - 1 = 0$$

and let (x, y), (x', y'), (x'', y'') be three points either situated upon the curve or not; also let

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = (1,1), \qquad \frac{x'x''}{a^2} + \frac{y'y''}{b^2} - 1 = (2,3),$$

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - 1 = (2,2), \qquad \frac{x''x}{a^2} + \frac{y''y}{b^2} - 1 = (3,1),$$

$$\frac{x''^2}{a^2} + \frac{y''^2}{b^2} - 1 = (3,3), \qquad \frac{xx'}{a^2} + \frac{yy'}{a^2} - 1 = (1,2),$$

then writing

$$|\,1,2,3\,| = \begin{vmatrix} \frac{x}{a} & \frac{y}{b} & 1 \\ \frac{x'}{a} & \frac{y'}{b} & 1 \\ \frac{x''}{a} & \frac{y''}{b} & 1 \end{vmatrix} = \pm 2\frac{A}{ab} \qquad |\,1,2,3\,|' = \begin{vmatrix} \frac{x}{a} & \frac{y}{b} & -1 \\ \frac{x'}{a} & \frac{y'}{b} & -1 \\ \frac{x''}{a} & \frac{y''}{b} & -1 \end{vmatrix} = \mp 2\frac{A}{ab}$$

where A is the area of the triangle whose angular points are at (x, y), (x', y'), (x'', y''), there results

$$|\,1,2,3\,|\;|\,1,2,3\,|' = -\frac{4A^2}{a^2b^2}$$

but by the theorem of the present section

$$|\,1,2,3\,|\;|\,1,2,3\,|' = \begin{vmatrix} (1,1) & (1,2) & (1,3) \\ (2,1) & (2,2) & (2,3) \\ (3,1) & (3,2) & (3,3) \end{vmatrix}$$

and consequently

$$A = \tfrac{1}{2}ab\{-(1,1)(2,2)(3,3) + (1,1)(2,3)^2 + (2,2)(3,1)^2 + (3,3)(1,2)^2 - 2(2,3)(3,1)(1,2)\}^{\frac{1}{2}}$$

This formula comprises a large number of theorems.

When the triangle is inscribed in the conic

$$(1,1) = 0, \quad (2,2) = 0, \quad (3,3) = 0$$

and if f, g, h be the chords joining the points two and two, and F, G, H the semi-diameters respectively parallel to f, g, h,

$$-2(2,3) = \frac{(x'-x'')^2}{a^2} + \frac{(y'-y'')^2}{b^2} = \frac{f^2}{F^2}$$

$$-2(3,1) = \frac{(x''-x)^2}{a^2} + \frac{(y''-y)^2}{b^2} = \frac{g^2}{G^2}$$

$$-2(1,2) = \frac{(x-x')^2}{a^2} + \frac{(y-y')^2}{b^2} = \frac{h^2}{H^2}$$

and consequently

$$A = \tfrac{1}{4}ab\frac{fgh}{FGH}$$

which expresses, that, twice the area of a triangle inscribed in an ellipse is to the product of the principal axes as the product of the sides is to the product of the diameters parallel to them.

If the ellipse becomes a circle,

$$a = b = F = G = H = r$$

where r is the radius of the circle, and consequently

$$A = \frac{fgh}{4r}$$

and dividing this by the corresponding equation in the ellipse,

$$r = \frac{FGH}{ab}$$

and consequently

> The radius of a circle which passes through three points on an ellipse is equal to the product of the semi-diameters parallel to the sides of the inscribed triangle, divided by the product of the semi-axes.

The equation to the conic, when referred to one of its foci as the origin, is

$$x^2 + y^2 - \lambda^2(x+p) = 0$$

And if u, v, w be the three focal chords parallel to the three sides of an inscribed triangle, s another focal chord perpendicular to the major axis, and r the radius of the circle passing through the angular points of the inscribed triangle, there would be found, by a process similar to that used above,

$$r = \frac{1}{2}\sqrt{\left(\frac{uvw}{s}\right)}$$

In the general formula given above, when the three points are conjugate, that is to say, when the polar of each passes through the other two, we have

$$(2,3) = 0, \quad (3,1) = 0, \quad (1,2) = 0$$

and the expression $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^{\frac{1}{2}}$ is equal to the distance of any point from the centre of the conic, divided by the semi-diameter parallel to that distance; so that if e, e', e'' be the distances of the three conjugate points from the centre, and d, d', d'' the semi-diameters respectively parallel to them, the area of the triangle will be given by the equation

$$\mathrm{A} = \frac{1}{2}ab\sqrt{\left(-\left(\frac{e^2}{d^2}-1\right)\left(\frac{e'^2}{d'^2}-1\right)\left(\frac{e''^2}{d''^2}-1\right)\right)}$$

§ VI.—*On inverse Systems, and Determinants of Determinants.*

Consider the linear equations

$$\begin{array}{l} (1,1)x_1 + (1,2)x_2 + \cdots + (1,n)x_n = u_1 \\ (2,1)x_1 + (2,2)x_2 + \cdots + (2,n)x_n = u_2 \\ \quad \cdot \qquad\quad \cdot \qquad\quad \cdot\cdot \qquad \cdot \qquad \cdot \\ (n,1)x_1 + (n,2)x_2 + \cdot + (n,n)x_n = u_n \end{array}$$

the solutions of which may be thus written,

$$\begin{array}{l}
|\,1,2,\cdot\cdot\, n\,|\; x_1 = [1,1]u_1 + [1,2]u_2 + \cdot\cdot + [1,n]u_n \\
|\,1,2,\cdot\cdot\, n\,|\; x_2 = [2,1]u_1 + [2,2]u_2 + \cdot\cdot + [2,n]u_n \\
\quad\cdot \qquad \cdot \qquad \cdot \qquad \cdot\cdot \qquad \cdot \\
|\,1,2,\cdot\cdot\, n\,|\; x_n = [n,1]u_1 + [n,2]u_2 + \cdot\cdot + [n,n]u_n
\end{array}$$

where the values of [1,1], [1,2], . . are known by what is gone before, being in fact the coefficients of (1,1), (1,2), . . in the development of | 1,2, . . n | when the rows in which those quantities stand are respectively taken first in the development. The functions [1,1], [1,2] . . possess some remarkable and useful properties, which it is now proposed to investigate.

In the first place it is seen by multiplying the last, or, as it will generally be called, *inverse* system of equations by the factors

$$\begin{array}{cccc}
(1,1), & (1,2,) & \cdot\cdot & (1,n) \\
(2,1), & (2,2), & \cdot\cdot & (2,n) \\
\cdot & \cdot & \cdot\cdot & \cdot \\
(n,1), & (n,2), & \cdot\cdot & (n,n)
\end{array}$$

that the following relations are produced

$$\begin{array}{llll}
(1,1)[1,1]+(1,2)[2,1]+\cdot\cdot=\nabla, & (1,1)[1,2]+(1,2)[2,2]+\cdot\cdot=0, & \cdot\cdot & (1,1)[1,n]+(1,2)[2,n]+\cdot\cdot=0 \\
(2,1)[1,1]+(2,2)[2,1]+\cdot\cdot=0, & (2,1)[1,2]+(2,2)[2,2]+\cdot\cdot=\nabla, & \cdot\cdot & (2,1)[1,n]+(2,2)[2,n]+\cdot\cdot=0 \\
\cdot \qquad \cdot & \cdot \qquad \cdot & \cdot\cdot & \cdot \qquad \cdot \\
(n,1)[1,1]+(n,2)[2,1]+\cdot\cdot=0, & (n,1)[1,2]+(n,2)[2,2]+\cdot\cdot=0, & \cdot\cdot & (n,1)[1,n]+(n,2)[2,n]+\cdot\cdot=\nabla
\end{array}$$

where

$$\nabla = |\, 1,\, 2,\, \cdot\cdot\, n\, |$$

and forming the determinant of the expressions on the left hand sides of these equations, and also that of those on the right hand sides, and equating the results, it is found

$$\left|\begin{array}{lll}
(1,1)[1,1]+(1,2)[2,1]+\cdot\cdot & (1,1)[1,2]+(1,2)[2,2]+\cdot\cdot & \cdot\cdot\; (1,1)[1,n]+(1,2)[2,n]+\cdot\cdot \\
(2,1)[1,1]+(2,2)[2,1]+\cdot\cdot & (2,1)[1,2]+(2,2)[2,2]+\cdot\cdot & \cdot\cdot\; (2,1)[1,n]+(2,2)[2,n]+\cdot\cdot \\
\cdot \qquad \cdot & \cdot \qquad \cdot & \cdot\cdot \quad \cdot \qquad \cdot \\
(n,1)[1,1]+(n,2)[2,1]+\cdot\cdot & (n,1)[1,2]+(n,2)[2,2]+\cdot\cdot & \cdot\cdot\; (n,1)[1,n]+(2,2)[2,n]+\cdot\cdot
\end{array}\right|$$

$$= \left|\begin{array}{cccc}
\nabla & \ast & \cdot\cdot & \ast \\
\ast & \nabla & \cdot & \ast \\
\cdot & \cdot & & \cdot \\
\ast & \ast & \cdot\cdot & \nabla
\end{array}\right|$$

or by Theorem XI.

$$\begin{vmatrix} (1,1) & (1,2) & \cdots & (1,n) \\ (2,1) & (2,2) & \cdots & (2,n) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (n,1) & (n,2) & \cdots & (n,n) \end{vmatrix} \begin{vmatrix} [1,1] & [1,2] & \cdots & [1,n] \\ [2,1] & [2,2] & \cdots & [2,n] \\ \cdot & \cdot & \cdot\cdot & \cdot \\ [n,1] & [n,2] & \cdots & [n,n] \end{vmatrix} = \nabla^n$$

i. e.

$$\begin{vmatrix} [1,1] & [1,2] & \cdots & [1,n] \\ [2,1] & [2,2] & \cdots & [2,n] \\ \cdot & \cdot & \cdot\cdot & \cdot \\ [n,1] & [n,2] & \cdots & [n,n] \end{vmatrix} = \nabla^{n-1}$$

A more general theorem may, however, be deduced as follows: when

$$u_1 = 0, \; u_2 = 0, \; \cdots \; u_i = 0,$$

the latter equations give rise to the following,

$$\begin{array}{l} [i+1,i+1]u_{i+1} + [i+1,i+2]u_{i+2} + \cdots + [i+1,n]u_n = |\,1,2, \cdots n\,|\, x_{i+1} \\ [i+2,i+1]u_{i+1} + [i+2,i+2]u_{i+2} + \cdots + [i+2,n]u_n = |\,1,2, \cdots n\,|\, x_{i+2} \\ \quad \cdot \qquad\qquad\qquad \cdot \qquad\qquad \cdot\cdot \qquad \cdot \qquad\qquad \cdot \qquad \cdot \\ [n,i+1]u_{i+1} \quad + \quad [n,i+2]u_{i+2} \quad + \cdots + \quad [n,n]u_n \quad = |\,1,2, \cdots n\,|\, x_n \end{array}$$

Whence also the inverse system,

$$\begin{vmatrix} [i+1,i+1] & [i+1,i+2] & \cdots & [i+1,n] \\ [i+2,i+1] & [i+2,i+2] & \cdots & [i+2,n] \\ \cdot & \cdot & \cdot\cdot & \cdot \\ [n,i+1] & [n,i+2] & \cdots & [n,n] \end{vmatrix} u_{i+1}$$

$$= |\,1,2, \cdots n\,| \begin{vmatrix} [i+2,i+2] & [i+2,i+3] & \cdots & [i+2,n] \\ [i+3,i+2] & [i+3,i+3] & \cdots & [i+3,n] \\ \cdot & \cdot & \cdot\cdot & \cdot \\ [n,i+2] & [n,1+3] & \cdots & [n,n] \end{vmatrix} x_{i+1} + \cdots$$

Now writing the first i equations of the given system thus,

$$\begin{array}{l} (1,1)x_1 + (1,2)x_2 + \cdots + (1,i)x_i = -(1,i+1)x_{i+1} - \cdots - (1,n)x_n \\ (2,1)x_1 + (2,2)x_2 + \cdots + (2,i)x_i = -(2,i+1)x_{i+1} - \cdots - (2,n)x_n \\ \quad \cdot \qquad\quad \cdot \qquad \cdot\cdot \qquad \cdot \qquad\qquad \cdot \qquad\quad \cdot\cdot \qquad \cdot \\ (i,1)x_1 + (i,2)x_2 + \cdots + (i,i)x_i = -(i,i+1)x_{i+1} - \cdots - (i,n)x_n \end{array}$$

there may be deduced,

$$\begin{vmatrix} (1,1) & (1,2) & \cdot\cdot & (1,i) \\ (2,1) & (2,2) & \cdot\cdot & (2,i) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (i,1) & (i,2) & \cdot\cdot & (i,i) \end{vmatrix} x_1 = - \begin{vmatrix} (1,i+1) & (1,2) & \cdot\cdot & (1,i) \\ (2,i+1) & (2,2) & \cdot\cdot & (2,i) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (i,i+1) & (i,2) & \cdot\cdot & (i,i) \end{vmatrix} x_{i+1} - \cdot\cdot$$

$$\vdots$$

$$\begin{vmatrix} (1,2) & (1,3) & \cdot\cdot & (1,1) \\ (2,2) & (2,3) & \cdot\cdot & (2,1) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (i,2) & (i,3) & \cdot\cdot & (i,1) \end{vmatrix} x_2 = - \begin{vmatrix} (1,i+1) & (1,3) & \cdot\cdot & (1,1) \\ (2,i+1) & (2,3) & \cdot\cdot & (2,1) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (i,i+1) & (i,3) & \cdot\cdot & (i,1) \end{vmatrix} x_{i+1} - \cdot\cdot$$

$$\cdot$$

$$\begin{vmatrix} (1,i) & (1,1) & \cdot\cdot & (1,i-1) \\ (2,i) & (2,1) & \cdot\cdot & (2,i-1) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (i,i) & (i,1) & \cdot\cdot & (i,i-1) \end{vmatrix} x_i = - \begin{vmatrix} (1,i+1) & (1,1) & \cdot\cdot & (1,i-1) \\ (2,i+1) & (2,1) & \cdot\cdot & (2,i-1) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (i,i+1) & (i,1) & \cdot\cdot & (i,i-1) \end{vmatrix} x_{i+1} - \cdot\cdot$$

or

$$|\,1,2,\cdot\cdot\, i\,|\, x_1 = \pm \begin{vmatrix} (1,2) & (1,3) & \cdot\cdot & (1,i+1) \\ (2,2) & (2,3) & \cdot\cdot & (2,i+1) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (i,2) & (i,3) & \cdot\cdot & (i,i+1) \end{vmatrix} x_{i+1} \cdot\cdot,$$

$$|\,1,2,\cdot\cdot\, i\,|\, x_2 = \begin{vmatrix} (1,3) & (1,4) & \cdot\cdot & (1,1) \\ (2,3) & (2,4) & \cdot\cdot & (2,1) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (i,3) & (i,4) & \cdot\cdot & (i,i) \end{vmatrix} x_{i+1} \cdot\cdot,$$

$$|\,1,2,\cdot\cdot\, i\,|\, x_i = \pm \begin{vmatrix} (1,i+1) & (1,1) & \cdot\cdot & (1,i-1) \\ (2,i+1) & (2,1) & \cdot\cdot & (2,i-1) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (i,i+1) & (i,1) & \cdot\cdot & (i,i-1) \end{vmatrix} x_{i+1} \cdot\cdot$$

so that

$$\begin{aligned} u_{i+1} &= (i+1,1)x_1 + (i+1,2)x_2 + \cdot\cdot + (i+1,i+1)x_{i+1} + \cdot\cdot \\ &= |\,1,2,\cdot\cdot\, i\,|^{-1}\, |\,1,2,\cdot\cdot\, i+1\,|\, x_{i+1} + \cdot\cdot \end{aligned}$$

and consequently, substituting in the former expression for u_{i+1} and equating the coefficients of x_{i+1}

$$\begin{vmatrix} [i+1,i+1] & [i+1,i+2] & \cdot\cdot & [i+1,n] \\ [i+2,i+1] & [i+2,i+2] & \cdot\cdot & [i+2,n] \\ \cdot & \cdot & \cdot\cdot & \\ [n,i+1] & [n,i+2] & \cdot\cdot & [n,n] \end{vmatrix}$$

$$= \begin{vmatrix} [i+2,i+2] & [i+2,i+3] & \cdot\cdot & [i+2,n] \\ [i+3,i+2] & [i+3,i+3] & \cdot\cdot & [i+3,n] \\ \cdot & \cdot & \cdot\cdot & \cdot \\ [n,i+2] & [n,i+3] & \cdot\cdot & [n,n] \end{vmatrix} |1,2,\cdot\cdot n|\,|1,2,\cdot\cdot i|\,|1,2,\cdot\cdot i+1|^{-1}$$

and similarly performing this method of reduction (j) times, it would at length be found that

$$\begin{vmatrix} [i+1,i+1] & [i+1,i+2] & \cdot\cdot & [i+1,n] \\ [i+2,i+1] & [i+2,i+2] & \cdot\cdot & [i+2,n] \\ \cdot & \cdot & \cdot\cdot & \cdot \\ [n,i+1] & [n,i+2] & \cdot\cdot & [n,n] \end{vmatrix}$$

$$= \begin{vmatrix} [i+j+1,i+j+1] & [i+j+1,i+j+2] & \cdot\cdot & [i+j+1,n] \\ [i+j+2,i+j+1] & [i+j+2,i+j+2] & \cdot\cdot & [i+j+2,n] \\ \cdot & \cdot & \cdot\cdot & \\ [n,i+j+1] & [n,i+j+2] & \cdot\cdot & [n,n] \end{vmatrix} \begin{matrix} |1,2,\cdot\cdot n|^{j}\,|1,2,\cdot\cdot i| \\ \times |1,2,\cdot\cdot i+j|^{-1} \end{matrix}$$

This formula includes as a particular case M. Jacobi's theorem, viz. if

$$i+j+1=n$$

then

$$\begin{vmatrix} [i+1,i+1] & [i+1,i+2] & \cdot\cdot & [i+1,n] \\ [i+2,i+1] & [i+2,i+2] & \cdot\cdot & [i+2,n] \\ \cdot & \cdot & \cdot\cdot & \cdot \\ [n,i+1] & [n,i+2] & \cdot\cdot & [n,n] \end{vmatrix} \begin{matrix} = |1,2,\cdot\cdot n|^{n-i-1}\,|\,|1,2,\cdot\cdot i|\,|1,2,\cdot\cdot n-1|^{-1}[n,n] \\ = |1,2,\cdot\cdot n|^{n-i-1}\;|1,2,\cdot\cdot i| \end{matrix}$$

As particular cases of the latter theorem the following may be noticed,

$$\begin{vmatrix} [n-1,n-1] & [n-1,n] \\ [n,n-1] & [n,n] \end{vmatrix} = |1,2,\cdot\cdot n|\;\;|1,2,\cdot\cdot n-2|$$

$$\begin{vmatrix} [n-2,n-2] & [n-2,n-1] & [n-2,n] \\ [n-1,n-2] & [n-1,n-1] & [n-1,n] \\ [n,n-2] & [n,n-1] & [n,n] \end{vmatrix} = |1,2,\cdot\cdot n|^{2}\;|1,2,\cdot\cdot n-3|$$

$$\cdot\;\;\cdot\;\;\cdot$$

$$\begin{vmatrix} [2,2] & [2,3] & \cdot\cdot & [2,n] \\ [3,2] & [3,3] & \cdot\cdot & [3,n] \\ \cdot & \cdot & \cdot\cdot & \cdot \\ [n,2] & [n,3] & \cdot\cdot & [n,n] \end{vmatrix} = |1,2,\cdot\cdot n|^{n-2}(1,1)$$

$$\begin{vmatrix} [1,1] & [1,2] & \cdot\cdot & [1,n] \\ [2,1] & [2,2] & \cdot\cdot & [2,n] \\ \cdot & \cdot & \cdot\cdot & \cdot \\ [n,1] & [n,2] & \cdot\cdot & [n,n] \end{vmatrix} = |1,2,\cdot\cdot n|^{n-1}$$

There is a large class of determinants whose constituents satisfy the conditions

$$\begin{matrix} * & (1,2)=(2,1), & \cdot\cdot & (1,n)=(n,1) \\ (2,1)=(1,2), & * & \cdot\cdot & (2,n)=(n,2) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (n,1)=(1,n), & (n,2)=(2,n), & \cdot\cdot & * \end{matrix}$$

and it is observable that by taking the system of conditions between (1,1), (1,2) . . , and [1,1], [1,2], . . given in the early part of this section, and omitting those lying upon the diagonal, there may be deduced

$$\begin{array}{lllcllll}
[1,1] & : [1,2] & : \cdot\cdot[1,n] & = & [1,1] & : [2,1] & : \cdot\cdot[n,1] \\
: [2,1] & : [2,2] & : \cdot\cdot[2,n] & & : [1,2] & : [2,2] & : \cdot\cdot[n,2] \\
. & . & \cdot\cdot\ . & & . & . & \cdot\cdot\ . \\
: [n,1] & : [n,2] & : \cdot\cdot[n,n] & & : [1,n] & : [2,n] & : \cdot\cdot[n,n]
\end{array}$$

or

$$\begin{array}{cccc}
* & [1,2] = [2,1], & \cdot\cdot & [1,n] = [n,1] \\
[2,1] = [1,2], & * & \cdot\cdot & [2,n] = [n,2] \\
. & . & \cdot\cdot & . \\
[n,1] = [1,n], & [n,2] = [2,n], & \cdot\cdot & *
\end{array}$$

It is perhaps worth while to write down the developments of the determinants belonging to this class for the cases $n=3$ and $n=4$. These are as follows; for $n=3$,

$$(1,1)(2,2)(3,3)-(1,1)(2,3)^2-(2,2)(3,1)^2-(3,3)(1,2)^2+2(2,3)(3,1)(1,2)$$

and for $n=4$,

$$\begin{array}{c}
(1,1)\,(2,2)\,(3,3)\,(4,4) \\
-(2,2)\,(3,3)\,(1,4)^2-(3,3)\,(1,1)\,(2,4)^2-(1,1)\,(2,2)\,(3,4)^2 \\
-(1,1)\,(4,4)\,(2,3)^2-(2,2)\,(4,4)\,(3,1)^2-(3,3)\,(4,4)\,(1,2)^2 \\
+\quad (2,3)^2\,(1,4)^2 \quad + \quad (3,1)^2\,(2,4)^2 \quad + \quad (1,2)^2\,(3,4)^2 \\
-2\{(1,2)\,(1,3)\,(4,2)\,(4,3)+(2,3)\,(2,1)\,(4,3)\,(4,1)+(3,1)(3,2)(4,1)(4,2)\} \\
+\ 2(1,1)\,(2,3)\,(2,4)\,(3,4) \\
+\ 2(2,2)\,(3,1)\,(3,4)\,(1,4) \\
+\ 2(3,3)\,(1,2)\,(1,4)\,(2,4) \\
+\ 2(4,4)\,(2,3)\,(3,1)\,(1,2)
\end{array}$$

The equation to a cone, or other surface of the second order, may now be expressed in the same form as that to the reciprocal cone; for, adopting a usual notation, let

$$\mathfrak{A} = [1,1], \quad \mathfrak{B} = [2,2], \quad \mathfrak{C} = [3,3]$$

$$\mathfrak{F} = [2,3] = [3,2], \quad \mathfrak{G} = [3,1] = [1,3], \quad \mathfrak{H} = [1,2] = [2,1]$$

the equation to the reciprocal cone will be

$$\mathfrak{A}\xi^2 + \mathfrak{B}\eta^2 + \mathfrak{C}\zeta^2 + 2(\mathfrak{F}\eta\zeta + \mathfrak{G}\zeta\xi + \mathfrak{H}\xi\eta) = 0$$

so that unless

$$\left|\begin{array}{ccc}
\mathrm{A} & \mathrm{H} & \mathrm{G} \\
\mathrm{H} & \mathrm{B} & \mathrm{F} \\
\mathrm{G} & \mathrm{F} & \mathrm{C}
\end{array}\right| = 0$$

a process similar to that employed in § III. will give for the equation to the cone reciprocal to that above given, i.e. to the original cone,

$$\begin{vmatrix} \mathfrak{A} & \mathfrak{H} & \mathfrak{G} & x \\ \mathfrak{H} & \mathfrak{B} & \mathfrak{F} & y \\ \mathfrak{G} & \mathfrak{F} & \mathfrak{C} & z \\ x & y & z & 0 \end{vmatrix} = 0$$

As an example, let it be proposed to investigate the conditions that the roots of the equation

$$\begin{vmatrix} (1,1)-\theta & (1,2) & \cdot\cdot & (1,n) \\ (2,1) & (2,2)-\theta & \cdot\cdot & (2,n) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (n,1) & (n,2) & \cdot\cdot & (n,n)-\theta \end{vmatrix} = 0$$

(in which it is supposed that $(1,2) = (2,1)$ . .) shall be all positive. By what has been shown before, this equation may be resolved into systems of the following form,

$$\begin{array}{l} (1,1)x_1 + (1,2)x_2 + \cdot\cdot + (1,n)x_n = \theta_1 x_1 \\ (2,1)x_1 + (2,2)x_2 + \cdot\cdot + (2,n)x_n = \theta_1 x_2 \\ \quad\cdot \qquad\quad \cdot \qquad \cdot\cdot \qquad \cdot \qquad\quad \cdot \\ (n,1)x_1 + (n,2)x_2 + \cdot\cdot + (n,n)x_n = \theta_1 x_n \end{array}$$

where θ_1 is a root of the given equation; the other systems being formed by writing, $y_1, y_2, \ldots y_n, \theta_2 \ldots$ successively for $x_1, x_2, \ldots x_n, \theta_1$. And if the systems of variables $x, y, \ldots$ be subject to the conditions

$$\begin{array}{l} x_1^2 + x_2^2 + \cdot\cdot + x_n^2 = 1 \\ y_1^2 + y_2^2 + \cdot\cdot + y_n^2 = 1 \\ \cdot \quad \cdot \quad \cdot\cdot \quad \cdot \end{array}$$

the following equations may be formed,

$$\begin{array}{l} (1,1)x_1^2 + \cdot\cdot + 2(1,2)x_1x_2 + \cdot\cdot = \theta_1 \\ (1,1)y_2^2 + \cdot\cdot + 2(1,2)y_1y_2 + \cdot\cdot = \theta_2 \\ \cdot \quad \cdot\cdot \quad \cdot \quad \cdot\cdot \quad \cdot \end{array}$$

$\theta_1, \theta_2, \ldots$ being the roots of the given equation. The following inverse systems may be easily formed;

$$\begin{array}{l} [1,1]x_1 + [1,2]x_2 + \cdot\cdot + [1,n]x_n = \Theta_1 x_1 \\ [2,1]x_1 + [2,2]x_2 + \cdot\cdot + [2,n]x_n = \Theta_1 x_2 \\ \quad\cdot \qquad\quad \cdot \qquad \cdot\cdot \qquad \cdot \qquad\quad \cdot \\ [n,1]x_1 + [n,2]x_2 + \cdot\cdot + [n,n]x_n = \Theta_1 x_n \end{array}$$

$$\begin{array}{l} [1,1]x_1^2 + \cdot\cdot + 2[1,2]x_1x_2 + \cdot\cdot = \Theta_1 \\ [1,1]y_1^2 + \cdot\cdot + 2[1,2]y_1y_2 + \cdot\cdot = \Theta_2 \\ \cdot \quad \cdot\cdot \quad \cdot \quad \cdot\cdot \quad \cdot \end{array}$$

where

$$\Theta_i = \frac{|\,1, 2, \ldots n\,|}{\theta_i}$$

and similarly, by using the expressions $[1,1]_i$, $[1,2]_i, \ldots$, $|\,1,2, \ldots i\,|$, there might be formed the inverse systems when $(n - i)$ of the variables are equated to zero, and i out of the n variables are selected for the transformation.

Now since one condition of the problem is obviously

$$|\,1, 2, \ldots n\,| > 0$$

it will be sufficient for the present purpose to determine the relations among the coefficients, in order that the above systems of quadratic functions, or those formed when i variables only are used, may remain positive for all values of the variables. And since the n variables $x_1, x_2, \ldots x_n$, are subject to only two conditions, $(n - 2)$ of them will remain independent, and if all the groups of $(n - 2)$ be successively equated to zero, there will result a series of conditions, of which the following is one,

$$(1,1)\, x_1^2 + (2,2)\, x_2^2 + 2(1,2)\, x_1 x_2 > 0$$

these give rise to a series of conditions among the coefficients, of which the following is one,

$$|\,1, 2\,| > 0$$

The inverse system in the case of two variables presents no new feature, but if the inverse system be formed with three variables, and one of them be then equated to zero, there will be formed with x_1, x_2, x_3, the following system of conditions,

$$[2,2]_3 x_2^2 + [3,3]_3 x_3^2 + 2[2,3]_3 x_2 x_3 > 0$$
$$[3,3]_3 x_3^2 + [1,1]_3 x_1^2 + 2[3,1]_3 x_3 x_1 > 0$$
$$[1,1]_3 x_1^2 + [2,2]_3 x_2^2 + 2[1,2]_3 x_1 x_2 > 0$$

whence also the following,

$$(1,1)\,|\,1,2,3\,| > 0, \quad (2,2)\,|\,1,2,3\,| > 0, \quad (3,3)\,|\,1,2,3\,| > 0$$

with similar conditions for all the other ternary combinations of the variables.

Again, with x_1, x_2, x_3, x_4, there would be formed six conditions, of which the following is one,

$$[1,1]_4 x_1^2 + [2,2]_4 x_2^2 + 2[1,2]_4 x_1 x_2 > 0$$

there would be thence deduced

$$|\,2,3\,|\,|\,1,2,3,4\,| > 0, \quad |\,3,1\,|\,|\,1,2,3,4\,| > 0, \quad |\,1,2\,|\,|\,1,2,3,4\,| > 0$$
$$|\,1,4\,|\,|\,1,2,3,4\,| > 0, \quad |\,2,4\,|\,|\,1,2,3,4\,| > 0, \quad |\,3,4\,|\,|\,1,2,3,4\,| > 0$$

But, on account of the conditions found in the case of two variables, these are equivalent to only one, viz.

$$| 1{,}2{,}3{,}4 | > 0$$

with similar conditions for all the other quaternary combinations of the variables.

Similarly, the next series of conditions would contain the following,

$$| 1{,}2{,}3 | \; | 1{,}2{,}3{,}4{,}5 | > 0$$

which, by what has gone before, is equivalent to

$$(1{,}1) \, | 1{,}2{,}3{,}4{,}5 | > 0$$

and similarly for all the other conditions belonging to this group. And hence generally, i being any positive whole number, there will be a series of conditions, of which the following form a pair,

$$| 1{,}2, \cdot\cdot\, 2i | > 0$$
$$(1{,}1) \, | 1{,}2, \cdot\cdot\, 2i \pm 1 | > 0$$

the number of conditions in these cases will be respectively

$$\frac{n(n-1) \cdot\cdot (n-2i+1)}{1{,}2 \cdot\cdot (2i)}$$

and

$$(2i \pm 1)\frac{n(n-1) \cdot\cdot (n-(2i \pm 1)+1)}{1{,}2 \cdot\cdot (2i \pm 1)}$$

and the whole system of conditions may consequently be comprised under the two general formulæ,

$$\begin{vmatrix} (k_1, k_1) & (k_1, k_2) & \cdot\cdot & (k_1, k_{2i}) \\ (k_2, k_1) & (k_2, k_2) & \cdot\cdot & (k_2, k_{2i}) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (k_{2i}, k_1) & (k_{2i}, k_2) & \cdot\cdot & (k_{2i}, k_{2i}) \end{vmatrix} > 0, \quad (k_j, k_j) \begin{vmatrix} (k_1, k_1) & (k_1, k_2) & \cdot\cdot & (k_1, k_{2i\pm1}) \\ (k_2, k_1) & (k_2, k_2) & \cdot\cdot & (k_2, k_{2i\pm1}) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (k_{2i\pm1}, k_1) & (k_{2i\pm1}, k_2) & \cdot\cdot & (k_{2i\pm1}, k_{2i\pm1}) \end{vmatrix} > 0$$

j being any number comprised between the limits 1 and $(2i \pm 1)$ inclusively.

§ VII.—*Expressions for a Determinant and its Constituents in Terms of its differential Coefficients.*

It appears from the preceding section that a determinant may be expressed in any of the following forms,

$$\begin{aligned} \nabla &= (1{,}1)\,[1{,}1] + (1{,}2)\,[2{,}1] + \cdot\cdot + (1{,}n)\,[n{,}1] \\ &= (2{,}1)\,[1{,}2] + (2{,}2)\,[2{,}2] + \cdot\cdot + (2{,}n)\,[n{,}2] \\ &= \quad \cdot \qquad \cdot \qquad \cdot\cdot \qquad \cdot \\ &= (n{,}1)\,[1{,}n] + (n{,}2)\,[2{,}n] + \cdot\cdot + (n{,}n)\,[n{,}n] \end{aligned}$$

and consequently

$$\begin{array}{llll}
[1,1] = \dfrac{d\nabla}{d(1,1)}, & [1,2] = \dfrac{d\nabla}{d(2,1)}, & \cdot\cdot\ [1,n] = \dfrac{d\nabla}{d(n,1)}, \\
[2,1] = \dfrac{d\nabla}{d(1,2)}, & [2,2] = \dfrac{d\nabla}{d(2,2)}, & \cdot\cdot\ [2,n] = \dfrac{d\nabla}{d(n,2)} \\
\cdot & \cdot & \cdot\cdot \quad \cdot \\
[n,1] = \dfrac{d\nabla}{d(1,n)}, & [n,2] = \dfrac{d\nabla}{d(2,n)}, & \cdot\cdot\ [n,n] = \dfrac{d\nabla}{d(n,n)}
\end{array}$$

so that $D\nabla$ may be expressed as follows,

$$\begin{aligned}
D\nabla &= [1,1]\, d(1,1) + [2,1]\, d(1,2) + \cdot\cdot + [n,1]\, d(1,n) \\
&= [1,2]\, d(2,1) + [2,2]\, d(2,2) + \cdot\cdot + [n,2]\, d(2,n) \\
&= \quad \cdot \qquad\quad \cdot \qquad \cdot\cdot \qquad \cdot \\
&= [1,n]\, d(n,1) + [2,n]\, d(n,2) + \cdot\cdot + [n,n]\, d(n,n)
\end{aligned}$$

Inversely also,

$$\begin{array}{llll}
(1,1) = \dfrac{d\nabla}{d[1,1]}, & (1,2) = \dfrac{d\nabla}{d(2,1)}, & \cdot\cdot\ (1,n) = \dfrac{d\nabla}{d(n,1)} \\
(2,1) = \dfrac{d\nabla}{d[1,2]}, & (2,2) = \dfrac{d\nabla}{d[2,2]}, & \cdot\cdot\ (2,n) = \dfrac{d\nabla}{d[n,2]} \\
\cdot & \cdot & \cdot\cdot \quad \cdot \\
(n,1) = \dfrac{d\nabla}{d[1,n]}, & (n,2) = \dfrac{d\nabla}{d[2,n]}, & \cdot\cdot\ (n,n) + \dfrac{d\nabla}{d[n,n]}
\end{array}$$

$$\begin{aligned}
D\nabla &= (1,1)\, d[1,1] + (2,1)\, d[1,2] + \cdot\cdot + (n,1)\, d[1,n] \\
&= (1,2)\, d[2,1] + (2,2)\, d[2,2] + \cdot\cdot + (n,2)\, d[2,n] \\
& \quad \cdot \qquad\quad \cdot \qquad \cdot\cdot \qquad \cdot \\
&= (1,n)\, d[n,1] + (2,n)\, d[n,2] + \cdot\cdot + (n,n)\, d[n,n]
\end{aligned}$$

and similarly for the coefficients (2,1), (2,2), . . (3,1), (3,2), ...

By means of these properties a certain class of linear equations may be reduced to a remarkable form.

The solutions of the equations

$$\begin{array}{l}
(1,1)x_1 + (1,2)x_2 + \cdot\cdot + (1,n)x_n = u_1 \\
(1,2)x_1 + (2,2)x_2 + \cdot\cdot + (2,n)x_n = u_2 \\
\quad \cdot \qquad\quad \cdot \qquad \cdot\cdot \qquad \cdot \qquad \cdot \\
(1,n)x_1 + (2,n)x_2 + \cdot\cdot + (n,n)x_n = u_n
\end{array}$$

may be thus written,

$$\begin{array}{l}
\nabla x_1 = \frac{d\nabla}{d(1,1)}u_1 + \frac{d\nabla}{d(1,2)}u_2 + \cdot\cdot + \frac{d\nabla}{d(1,n)}u_n \\
\nabla x_2 = \frac{d\nabla}{d(1,2)}u_1 + \frac{d\nabla}{d(2,2)}u_2 + \cdot\cdot + \frac{d\nabla}{d(2,n)}u_n \\
\cdot \qquad \cdot \qquad \cdot \qquad \cdot\cdot \qquad \cdot \\
\nabla x_n = \frac{d\nabla}{d(1,n]}u_1 + \frac{d\nabla}{d(2,n)}u_2 + \cdot\cdot + \frac{d\nabla}{d(n,n)}u_n
\end{array}$$

and if there be a series of systems like the above in which the unknown quantities are

$$\begin{array}{cccc}
x_{1,1} & x_{1,2} & \cdot\cdot & x_{1,n} \\
x_{2,1} & x_{2,2} & \cdot\cdot & x_{2,n} \\
\cdot & \cdot & \cdot\cdot & \cdot \\
x_{n,1} & x_{n,2} & \cdot\cdot & x_{n,n}
\end{array}$$

respectively, the coefficients remaining the same, and the second members of the systems being

$$\begin{array}{llll}
\delta(1,1), & \delta(2,2) + ((1,2)), & \cdot\cdot & \delta(n,n) + ((1,n)) \\
\delta(1,1) - ((1,2)), & \delta(2,2), & \cdot\cdot & \delta(n,n) + ((2,n)) \\
\cdot & \cdot & \cdot\cdot & \cdot \\
\delta(1,1) - ((1,n)), & \delta(2,2) - ((2,n)), & \cdot\cdot & \delta(n,n)
\end{array}$$

then

$$\begin{array}{l}
\nabla x_{1,1} = \frac{d\nabla}{d(1,1)}\delta(1,1) + \frac{d\nabla}{d(1,2)}\delta(2,2) + \cdot\cdot + \frac{d\nabla}{d(1,n)}\delta(n,n) + * + \frac{d\nabla}{d(1,2)}((1,2)) + \cdot\cdot + \frac{d\nabla}{d(1,n)}((1,n)) \\
\nabla x_{2,2} = \frac{d\nabla}{d(1,2)}\delta(1,1) + \frac{d\nabla}{d(2,2)}\delta(2,2) + \cdot\cdot + \frac{d\nabla}{d(2,n)}\delta(n,n) - \frac{d\nabla}{d(1,2)}((1,2)) + * \cdot\cdot + \frac{d\nabla}{d(2,n)}((2,n)) \\
\cdot \qquad \cdot \qquad \cdot \qquad \cdot\cdot \\
\nabla x_{n,n} = \frac{d\nabla}{d(1,n)}\delta(1,1) + \frac{d\nabla}{d(2,n)}\delta(2,2) + \cdot\cdot + \frac{d\nabla}{d(n,n)}\delta(n,n) - \frac{d\nabla}{d(1,n)}((1,n)) - \frac{d\nabla}{d(2,n)}((2,n)) \cdot\cdot + *
\end{array}$$

whence

$$\nabla(x_{1,1} + x_{2,2} + \cdot\cdot + x_{n,n}) = \delta\nabla$$

or

$$x_{1,1} + x_{2,2} + \cdot\cdot\ x_{n,n} = \delta \log \nabla$$

where

$$\delta\nabla = \Sigma_i\Sigma_j \frac{d\nabla}{\delta(i,j)}\delta(i,j)$$

The following theorem, given by M. Malmstèn, will exemplify the use of determinants and the notation above adopted.

Let it be required to find the nth particular integral of the equation

$$(0,n) + \mathrm{P}(0,n-1) + \cdot\cdot + \mathrm{T}(0,0) = 0$$

where

$$(0,0) = y, \quad (0,1) = \frac{dy}{dx}, \cdot\cdot\ (0,n)\,\frac{d^ny}{dx^n}$$

when $(n-1)$ particular integrals

$$(1,0), \quad (2,0), \cdot\cdot\ (n,0)$$

are known. Suppose that

$$(0,0) = (1,0)k_1 + (2,0)k_2 + \cdot\cdot + (n-1,0)k_{n-1}$$

where k_1, k_2, .. k_{n-1} are to be so determined that the above value of $(0,0)$ shall satisfy the given equation. Suppose then, moreover, that

$$\begin{array}{l}
(1,0)k'_1 + (2,0)k'_2 + \cdot\cdot + (n-1,0)k'_{n-1} = 0 \\
(1,1)k'_1 + (2,1)k'^2 + \cdot\cdot + (n-1,1)k'_{n-1} = 0 \\
\quad . \qquad\quad . \qquad\quad .. \qquad\quad . \\
(1,n-3)k'_1 + (2,n-3)k'_2 + \cdot\cdot + (n-1,n-3)k'_{n-1} = 0
\end{array}$$

the solutions of which are

$$k'_1 : k'_2 : \cdot\cdot\ k'^{n-1}$$

$$= \begin{vmatrix} (2,0) & (3,0) & \cdot\cdot & (n-1,0) \\ (2,1) & (3,1) & \cdot\cdot & (n-1,1) \\ . & . & .. & . \\ (2,n-3) & (3,n-3) & \cdot\cdot & (n-1,\ n-3) \end{vmatrix} : \pm \begin{vmatrix} (3,0) & (4,0) & \cdot\cdot & (1,0) \\ (3,1) & (4,1) & \cdot\cdot & (1,1) \\ . & . & .. & . \\ (3,n-3) & (4,n-3) & \cdot\cdot & (1,n-3) \end{vmatrix} : \cdot\cdot \pm \begin{vmatrix} (1,0) & (2,0) & \cdot\cdot & (n-2,0) \\ (1,1) & (2,1) & \cdot\cdot & (n-2,1) \\ . & . & .. & . \\ (1,n-3) & (2,n-3) & \cdot\cdot & (n-2,n-3) \end{vmatrix}$$

$$= \mathrm{K}_1 : \mathrm{K}_2 : \cdot\cdot\ \mathrm{K}_{n-1}$$

suppose. On the other hand, by differentiating the expression for $(0,0)$, we find

$$\begin{array}{l}
(0,1) = (1,1)k_1 + (2,1)k_2 + \cdot\cdot + (n-1,1)k_{n-1} \\
(0,2) = (1,2)k_1 + (2,2)k_2 + \cdot\cdot + (n-1,2)k_{n-1} \\
\quad . \qquad\quad . \qquad\quad . \qquad\quad .. \qquad . \\
(0,n-1) = (1,n-1)k_1 + (2,n-1)k_2 + \cdot\cdot + (n-1,n-1)k_{n-1} \\
\qquad\qquad + (1,n-2)k'_1 + (2,n-2)k'_2 + \cdot\cdot + (n-1,n-2)k'_{n-1}
\end{array}$$

$$\begin{array}{l}
(0,n) = (1,n)k_1 + (2,n)k_2 + \cdot\cdot + (n-1,n)k_{n-1} \\
\qquad +2\{(1,n-1)k'_1 + (2,n-1)k'_2 + \cdot\cdot + (n-1,n-1)k'_{n-1}\} \\
\qquad + (1,n-2)k''_1 + (2,n-2)k''_2 + \cdot\cdot + (n-1,n-2)k''_{n-1}
\end{array}$$

Substituting these values in the given equation, there results

$$\begin{array}{l}
(1,n-2)k''_1+(2,n-2)k''_2+\cdot\cdot+(n-1,n-2)k''_{n-1}+\{\mathrm{P}(1,n-2)+2(1,n-1)\}k'_1 \\
\quad +\{\mathrm{P}(2,n-2)+2(2,n-1)\}k'_2+\cdot\cdot+\{\mathrm{P}(n-1,n-2)+2(n-1,n-1)\}k'_{n-1}=0
\end{array}$$

but from the values of $k'_1 : k'_2 : \dots k'_{n-1}$ found above, and writing

$$k'_1 = \theta \mathrm{K}_1, \quad k'_2 = \theta \mathrm{K}_2, \quad \cdot\cdot \quad k'_n = \theta \mathrm{K}_n$$

there follows

$$k''_1 = \theta' \mathrm{K}_1 + \theta \mathrm{K}'_1, \quad k''_2 = \theta' \mathrm{K}_2 + \theta \mathrm{K}'_2, \quad \cdot\cdot \quad k''_n = \theta' \mathrm{K}_n + \theta \mathrm{K}'_n$$

But

$$(1,n-2)\mathrm{K}_1 + (2,n-2)\mathrm{K}_2 + \cdot\cdot + (n-1,n-2)\mathrm{K}_{n-1}$$

$$= \begin{vmatrix} (1,0) & (2,0) & \cdot\cdot & (n-1,0) \\ (1,1) & (2,1) & \cdot\cdot & (n-1,1) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (1,n-2) & (2,n-2) & \cdot\cdot & (n-1,n-2) \end{vmatrix} = \nabla$$

and, as is easily seen,

$$\frac{d\nabla}{dx} = \begin{vmatrix} (1,0) & (2,0) & \cdot\cdot & (n-1,0) \\ (1,1) & (2,1) & \cdot\cdot & (n-1,1) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (1,n-3) & (2,n-3) & \cdot\cdot & (n-1,n-3) \\ (1,n-1) & (2,n-1) & \cdot\cdot & (n-1,n-1) \end{vmatrix}$$

Hence

$$(1,n-2)\mathrm{K}'_1 + (2,n-2)\mathrm{K}'_2 + \cdot\cdot + (n-1,n-2)\mathrm{K}'_{n-1} = 0$$

$$(1,n-1)\mathrm{K}_1 + (2,n-1)\mathrm{K}_2 + \cdot\cdot + (n-1,n-1)\mathrm{K}_{n-1} = \nabla'$$

So that the equation becomes

$$\theta'\nabla + \theta \mathrm{P} \nabla + 2\theta\nabla' = 0$$

or

$$\frac{\theta'}{\theta} + \frac{2\nabla'}{\nabla} + \mathrm{P} = 0$$

whence integrating

$$\theta\nabla^2 = e^{-\int \mathrm{P}dx}$$

or, substituting for θ in terms of k_i and K_i, and integrating again

$$k_i = \int \frac{\mathrm{K}_i}{\nabla^i} e^{-\int \mathrm{P}dx}\, dx$$

or writing

$$\Delta = \frac{1}{\nabla}, \qquad k_i = (-)^{n-1} \int \frac{d\,\Delta}{d(i,n-2)} e^{-\int \mathrm{P}dx}\, dx$$

Hence, if

$$y_1, y_2, \cdot\cdot\, y_{n-1}$$

be $(n-1)$ particular integrals of the equation

$$\frac{d^n y}{dx^n} + \mathrm{P}\frac{d^{n-1}y}{dx^{n-1}} + \cdot\cdot + \mathrm{T}y = 0$$

this equation will be also satisfied by,

$$y_n = k_1 y_1 + k_2 y_2 + \cdots + k_{n-1} y_{n-1}$$

where

$$k_i = (-)^{n-1} \int \frac{d\Delta}{dy_i^{(n-2)}} e^{-Pdx} dx$$

where $y^{(n)}$ is the nth differential coefficient of y with respect to x, and Δ has the value given above.

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§ VIII.—*On redundant Systems, and Groups of Determinants.*

The system

$$\begin{array}{c}
(1,1)x_1 + (1,2)x_2 + \cdots + (1,n)x_n = u_1 \\
(2,1)x_1 + (2,2)x_2 + \cdots + (2,n)x_n = u_2 \\
\cdot \quad \cdot \quad \cdot\cdot \quad \cdot \quad \cdot \\
(m,1)x_1 + (m,2)x_2 + \cdots + (m,n)x_n = u_m
\end{array}$$

where $m > n$, may be called a redundant system, there being more equations than necessary to determine the unknown quantities. There are, however, some remarkable formulæ connected with the solution of these equations, which may be here noticed. Suppose from the above system there be formed the following derived system,

$$\begin{array}{c}
(1,1)''x_1 + (1,2)x''_2 + \cdots + (1,n)''x_n = v_1 \\
(2,1)''x_1 + (2,2)''x_2 + \cdots + (2,n)''x_n = v_2 \\
\cdot \quad \cdot \quad \cdot\cdot \quad \cdot \quad \cdot \\
(n,1)''x_1 + (n,2)''x_2 + \cdots + (2,n)''x_n = v_n
\end{array}$$

where

$$\begin{array}{c}
(1,1)'u_1 + (1,2)'u_2 + \cdots + (1,m)'u_m = v_1 \\
(2,1)'u_1 + (2,2)'u_2 + \cdots + (2,m)'u_m = v_2 \\
\cdot \quad \cdot \quad \cdot\cdot \quad \cdot \quad \cdot \\
(n,1)'u_1 + (n,2)'u_2 + \cdots + (n,m)'u_m = v_n
\end{array}$$

so that the values of $(1,1)''$, $(1,2)''$, . . are obvious, being in fact identical with the constituents of the determinant discussed in § V.

Then every group of n equations out of the first system will give as usual

$$\begin{array}{c}
\nabla(x_1) = [1,1]u_1 + [1,2]u_2 + \cdots + [1,n]u_n \\
\nabla(x_2) = [2,1]u_1 + [2,2]u_2 + \cdots + [2,n]u_n \\
\cdot \quad \cdot \quad \cdot \quad \cdot\cdot \quad \cdot \\
\nabla(x_n) = [n,1]u_1 + [n,2]u_2 + \cdots + [n,n]u_n
\end{array}$$

where x_1, x_2, .. have been enclosed in parentheses to indicate that their values have been deduced from a redundant system.

And the derived system will give

$$\begin{array}{l}
\nabla'' x_1 = [1,1]'' v_1 + [1,2]'' v_2 + \cdots + [1,n]'' v_n \\
\nabla'' x_2 = [2,1]'' v_1 + [2,2]'' v_2 + \cdots + [2,n]'' v_n \\
\quad . \qquad\quad . \qquad\quad . \qquad\quad .. \qquad . \\
\nabla'' x_n = [n,1]'' v_1 + [n,2]'' v_2 + \cdots + [n,n]'' v_n
\end{array}$$

But if there be formed a series of partially derived systems, that is, systems derived by taking into account in succession the groups of n only out of the given equations, or, in other words, by putting $(m-n)$ of the quantities $u_1, u_2, .. u_m$, in turn equal to zero, the quantities ∇'', $[1,1]''$, $[1,2]''$, .. in each such system will be reduced simply to the determinants considered in § V.; and in fact, when the first n out of the given equations are taken into account, or, which is the same thing, when in the equations immediately above

$$u_{n+1} = 0, \quad u_{n+2} = 0, \ \cdots \ u_m = 0$$

then by the principles of § V.,

$$\nabla'' = \nabla\nabla'$$

$$\begin{array}{lll}
[1,1]'' = [1,1]\,[1,1]', & [1,2]'' = [1,2]\,[2,1]', \cdots & [1,n]'' = [1,n]\,[n,1]' \\
[2,1]'' = [2,1]\,[1,2]', & [2,2]'' = [2,2]\,[2,2]', \cdots & [2,n]'' = [2,n]\,[n,2]' \\
\qquad . & \qquad . \qquad\quad .. & \qquad . \\
[n,1]'' = [n,1]\,[1,n]', & [n,2]'' = [n,2]\,[2,n]', \cdots & [n,n]'' = [n,n]\,[n,n]'
\end{array}$$

so that the first group of equations will become

$$\begin{array}{l}
\nabla\nabla'(x_1) = [1,1]\,[1,1]' v_1 + [1,2]\,[2,1]' v_2 + \cdots + [1,n]\,[n,1]' v_n \\
\nabla\nabla'(x_2) = [2,1]\,[1,2]' v_1 + [2,2]\,[2,2]' v_2 + \cdots + [2,n]\,[n,2]' v_n \\
\quad . \qquad\qquad . \qquad\qquad . \qquad\qquad .. \qquad . \\
\nabla\nabla'(x_n) = [n,1]\,[1,n]' v_1 + [n,2]\,[2,n]' v_2 + \cdots + [n,n]\,[n,n]' v_n
\end{array}$$

Hence, summing all the corresponding equations of the various groups so formed, it is not difficult to see that by the principles of the addition and multiplication of determinants

$$\begin{array}{lll}
\Sigma[1,1]\,[1,1]' = [1,1]'', & \Sigma[1,2]\,[1,2]' = [1,2]'', \cdots & \Sigma[1,n]\,[1,n]' = [1,n]'' \\
\Sigma[2,1]\,[2,1]' = [2,1]'', & \Sigma[2,2]\,[2,2]' = [2,2]'', \cdots & \Sigma[2,n]\,[2,n]' = [2,n]'' \\
\qquad . & \qquad . \qquad\quad .. & \qquad . \\
\Sigma[n,1]\,[n,1]' = [n,1]'', & \Sigma[n,2]\,[n,2]' = [n,2]'', \cdots & \Sigma[n,n]\,[n,n]' = [n,n]''
\end{array}$$

$$\Sigma\nabla\nabla' = \nabla''$$

and consequently

$$x_1 = \frac{\Sigma\nabla\nabla'(x_1)}{\Sigma\nabla\nabla'}, \quad x_2 = \frac{\Sigma\nabla\nabla'(x_2)}{\Sigma\nabla\nabla'}, \cdot\cdot\ x_n = \frac{\Sigma\nabla\nabla'(x_n)}{\Sigma\nabla\nabla'}$$

The number of groups will be

$$m\frac{(m-1)\cdot\cdot(m-n+1)}{1,2\cdot\cdot n} = n\frac{(m-1)\cdot\cdot(n-1)}{1,2\cdot\cdot(m-n)}$$

Hence the following theorem may be enunciated:

Theorem XII. *If there be* m *linear equations involving* n *variables,* m *being* $>$ n, *or* m = n, *the values of the variables may be determined by solving the partially derived systems corresponding to each group of* n *equations, and dividing the sum of the values so found, each multiplied by its respective determinant, by the determinant of the completely derived system.*

A particular case of these equations is met with in the method of least squares; for let

$$\mathrm{U} = \Sigma\{(1,i)x_1 + (2,i)x_2 + \cdot\cdot + (n,i)x_n - u_i\}^2$$

the equations

$$\frac{1}{2}\frac{d\mathrm{U}}{dx_1} = 0, \quad \frac{1}{2}\frac{d\mathrm{U}}{dx_2} = 0, \quad \cdot\cdot\ \frac{1}{2}\frac{d\mathrm{U}}{dx_n} = 0$$

will in fact give

$$\begin{aligned} \Sigma(1,i)\{(1,i)x_1 + (2,i)x_2 + \cdot\cdot + (n,i)x_n - u_i\} &= 0 \\ \Sigma(2,i)\{(1,i)x_1 + (2,i)x_2 + \cdot\cdot + (n,i)x_n - u_i\} &= 0 \\ \cdot \qquad \cdot \qquad \cdot\cdot \qquad \cdot & \\ \Sigma(n,i)\{(1,i)x_1 + (2,i)x_2 + \cdot\cdot + (n,i)x_n - u_i\} &= 0 \end{aligned}$$

and will differ from those given above only in the conditions

$$\begin{aligned} &(1,1)' = (1,1), \quad (1,2)' = (1,2), \cdot\cdot\ (1,m)' = (1,m) \\ &(2,1)' = (2,1), \quad (2,2)' = (2,2), \cdot\cdot\ (2,m)' = (2,m) \\ &\quad \cdot \qquad\qquad \cdot \qquad\quad \cdot\cdot \qquad \cdot \\ &(m,1)' = (m,1), \quad (m,2)' = (m,2), \cdot\cdot\ (m,m)' = (m,m) \end{aligned}$$

and consequently also

$$\begin{aligned} &[1,1]' = [1,1], \quad [1,2]' = [1,2], \cdot\cdot\ [1,m]' = [1,m] \\ &[2,1]' = [2,1], \quad [2,2]' = [2,2], \cdot\cdot\ [2,m]' = [2,m] \\ &\quad \cdot \qquad\qquad \cdot \qquad\quad \cdot\cdot \qquad \cdot \\ &[m,1]' = [m,1], \quad [m,2]' = [m,2], \cdot\cdot\ [m,m]' = [m,m] \end{aligned}$$

and finally,

$$x_1 = \frac{\Sigma\nabla^2(x_1)}{\Sigma\nabla^2}, \quad x_2 = \frac{\Sigma\nabla^2(x_2)}{\Sigma\nabla^2}, \quad \cdot\cdot \quad x_n = \frac{\Sigma\nabla^2(x_n)}{\Sigma\nabla^2}$$

Hence also the following theorem may be enunciated:

> Theorem XIII. *If there be* m *linear equations involving* n *variables,* m *being* $>$n, *or* $=$ n, *and if the values of the variables deduced from each group of* n *equations be multiplied by the square of its corresponding determinant, the sum of all such quantities divided by the sum of the determinants will express the values of the variables deduced from the equations by the method of least squares.*

The square of the determinant corresponding to each group is called the *weight of the combination.*

Before quitting this subject there are one or two points which may be noticed. The solution of the equations arising from equating the partial differential coefficients of U to zero may be thus written,

$$\begin{aligned} \nabla' x_1 &= [1,1]' u'_1 + [1,2]' u'_2 + \cdot\cdot + [1,n]' u'_n \\ \nabla' x_2 &= [2,1]' u'_1 + [2,2]' u'_2 + \cdot\cdot + [2,n]' u'_n \\ &\cdot \quad \cdot \quad \cdot \quad \cdot\cdot \quad \cdot \\ \nabla' x_n &= [n,1]' u'_1 + [n,2]' u'_2 + \cdot\cdot + [n,n]' u'_n \end{aligned}$$

where

$$\begin{aligned} u' &= (1,1)u_1 + (1,2)u_2 + \cdot\cdot + (1,n)u_n \\ u'_2 &= (2,1)u_1 + (2,2)u_2 + \cdot\cdot + (2,n)u_n \\ &\cdot \quad \cdot \quad \cdot \quad \cdot\cdot \quad \cdot \\ u'_n &= (n,1)u_1 + (n,2)u_2 + \cdot\cdot + (n,n)u_n \end{aligned}$$

in which expressions the quantities

$$\frac{\nabla'}{[1,1]'} = \frac{\Sigma\nabla^2}{\Sigma[1,1]^2}, \quad \frac{\nabla'}{[2,2]'} = \frac{\Sigma\nabla^2}{\Sigma[2,2]^2}, \quad \cdot\cdot \quad \frac{\nabla'}{[n,n]'} = \frac{\Sigma\nabla^2}{\Sigma[n,n]^2}$$

are called the weights of the determinations of $x_1, x_2, \ldots x_n$. If only n observations be taken into account, the numerators of these expressions become simply ∇^2. If ε be the error to be feared in a determination whose weight is unity, the errors $E_1, E_2, \ldots E_n$ to be feared in the above determinations will be,

$$E_1 = \pm\,\varepsilon\sqrt{\frac{\Sigma[1,1]^2}{\Sigma\nabla^2}}, \quad E_2 = \pm\,\varepsilon\sqrt{\frac{\Sigma[2,2]^2}{\Sigma\nabla^2}}, \quad \cdot\cdot \quad E_n = \pm\,\varepsilon\sqrt{\frac{\Sigma[n,n]^2}{\Sigma\nabla^2}}$$

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§ IX.—*On Skew Determinants.*

A determinant whose constituents satisfy the conditions

$$\begin{array}{cccc} \ast & (1,2)+(2,1)=0, & \cdot\cdot & (1,n)+(n,1)=0 \\ (2,1)+(1,2)=0, & \ast & \cdot\cdot & (2,n)+(n,2)=0 \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (n,1)+(1,n)=0, & (n,2)+(2,n)=0, & \cdot\cdot & \ast \end{array}$$

is called *skew*. For the present it will be considered that

$$(1,1)=(2,2)=\cdot\cdot=(n,n)=1$$

Consider then the system

$$\begin{array}{ccccc} x_1 & +(1,2)x_2+ & \cdot\cdot & +(1,n)x_n & =u_1 \\ -(1,2)x_1+ & x_2 & +\cdot\cdot & +(2,n)x_n & =u_2 \\ \cdot & \cdot & \cdot\cdot & \cdot & \cdot \\ -(1,n)x_1 & -(2,n)x_2- & \cdot\cdot+ & x_n & =u_n \end{array}$$

and also the derived system

$$\begin{array}{ccccc} x_1 & -(1,2)x_2- & \cdot\cdot & -(1,n)x_n & =v_1 \\ (1,2)x_1+ & x^2 & -\cdot\cdot & -(2,n)x_n & =v_2 \\ \cdot & \cdot & \cdot\cdot & \cdot & \cdot \\ (1,n)x_1 & +(2,n)x_2+ & \cdot\cdot+ & x_n & =v_n \end{array}$$

and let

$$\begin{array}{llll} \nabla\{1,1\}=2[1,1]-\nabla, & \nabla\{1,2\}=2[1,2], & \cdot\cdot & \nabla\{1,n\}=2[1,n] \\ \nabla\{2,1\}=2[2,1], & \nabla\{2,2\}=2[2,2]-\nabla, & \cdot\cdot & \nabla\{2,n\}=2[2,n] \\ \cdot & \cdot & \cdot\cdot & \cdot \\ \nabla\{n,1\}=2[n,1], & \nabla\{n,2\}=2[n,2], & \cdot\cdot & \nabla\{n,n\}=2[n,n]-\nabla \end{array}$$

where

$$\nabla=\begin{vmatrix} 1 & (1,2) & \cdot\cdot & (1,n) \\ -(1,2) & 1 & \cdot\cdot & (2,n) \\ \cdot & \cdot & \cdot & \cdot \\ -(1,n) & -(2,n) & \cdot\cdot & 1 \end{vmatrix}$$

whence multiplying the given equations respectively by the factors

$$\begin{array}{cccc} \{1,1\}, & \{1,2\}, & \cdot\cdot & \{1,n\} \\ \{2,1\}, & \{2,2\}, & \cdot\cdot & \{2,n\} \\ \cdot & \cdot & \cdot\cdot & \cdot \\ \{n,1\}, & \{n,2\}, & \cdot\cdot & \{n,n\} \end{array}$$

and bearing in mind the relations given at the beginning of § VI. there results

$$\begin{array}{l}
\{1,1\}u_1 + \{1,2\}u_2 + \cdots + \{1,n\}u_n = v_1 \\
\{2,1\}u_1 + \{2,2\}u_2 + \cdots + \{2,n\}u_n = v_2 \\
\quad \cdot \qquad \cdot \qquad \cdot\cdot \qquad \vdots \\
\{n,1\}u_1 + \{n,2\}u_2 + \cdots + \{n,n\}u_n = v_n
\end{array}$$

and, similarly, from the derived system

$$\begin{array}{l}
\{1,1\}v_1 + \{2,1\}v_2 + \cdots + \{n,1\}v_n = u_1 \\
\{1,2\}v_1 + \{2,2\}v_2 + \cdots + \{n,1\}v_n = u_2 \\
\quad \cdot \qquad \cdot \qquad \cdot\cdot \qquad \cdot \qquad \cdot \\
\{1,n\}v_1 + \{2,n\}v_2 + \cdots + \{n,n\}v_n = u_n
\end{array}$$

so that

$$\begin{array}{lll}
\{1,1\}^2 + \{2,1\}^2 + \cdots \{n,1\}^2 = 1, & \{1,1\}\{1,2\} + \{2,1\}\{2,2\} + \cdots \{n,1\}\{n,2\} = 0, & \cdots \{1,1\}\{1,n\} + \{2,1\}\{2,n\} + \cdots \{n,1\}\{n,n\} = 0 \\
\{1,2\}\{1,1\} + \{2,2\}\{2,1\} + \cdots \{n,2\}\{n,1\} = 0, & \{1,2\}^2 + \{2,2\}^2 + \cdots \{n,2\}^2 = 1, & \cdots \{1,2\}\{1,n\} + \{2,2\}\{2,n\} + \cdots \{n,2\}\{n,n\} = 0 \\
\cdot & \cdot & \cdot\cdot \qquad \cdot \\
\{1,n\}\{1,1\} + \{2,n\}\{2,1\} + \cdots \{n,n\}\{n,1\} = 0, & \{1,n\}\{1,2\} + \{2,n\}\{2,2\} + \cdots \{n,n\}\{n,2\} = 0, & \cdots \{1,n\}^2 + \{2,n\}^2 + \cdots \{n,n\}^2 = 1
\end{array}$$

We have therefore found a system of n^2 quantities $\{1,1\}$, $\{1,2\}$ .., rational functions of $\frac{1}{2}n(n-1)$ independent variables, $(1,1)$, $(1,2)$ .. and satisfying the conditions given above.

The conditions above written give rise to a relation which is perhaps worth remarking; in fact, forming the determinant of the quantities on the left-hand side of the system immediately above; equating it to that formed from the quantities on the right-hand side, there will be found,

$$\left|\begin{array}{cccc}
\{1,1\} & \{1,2\} & \cdots & \{1,n\} \\
\{2,1\} & \{2,2\} & \cdots & \{2,n\} \\
\cdot & \cdot & \cdot\cdot & \cdot \\
\{n,1\} & \{n,2\} & \cdots & \{n,n\}
\end{array}\right|^2 = 1$$

or, substituting for $\{1,1\}$, $\{1,2\}$, .. their values, and extracting the square root,

$$\left|\begin{array}{cccc}
[1,1] - \frac{\nabla}{2} & [1,2] & \cdots & [1,n] \\
[2,1] & [2,2] - \frac{\nabla}{2} & \cdots & [2,n] \\
\cdot & \cdot & \cdot\cdot & \cdot \\
[n,1] & [n,2] & \cdots & [n,n] - \frac{\nabla}{2}
\end{array}\right| 2^n = \pm 1$$

As an example let $n = 3$, and

$$\nabla = \begin{vmatrix} 1 & \nu & -\mu \\ -\nu & 1 & \lambda \\ \mu & -\lambda & 1 \end{vmatrix} = 1 + \lambda^2 + \mu^2 + \nu^2$$

then, for the inverse system, we have

$$\begin{vmatrix} 1 + \lambda^2 & \lambda\mu + \nu & \nu\lambda - \mu \\ \lambda\mu - \nu & 1 + \mu^2 & \mu\nu + \lambda \\ \nu\lambda + \mu & \mu\nu - \lambda & 1 + \nu^2 \end{vmatrix}$$

and therefore,

$$\begin{array}{lll} \nabla\{1,1\} = 1 + \lambda^2 - \mu^2 - \nu^2, & \nabla\{1,2\} = 2(\lambda\mu + \nu), & \nabla\{1,3\} = 2(\nu\lambda - \mu) \\ \nabla\{2,1\} = 2(\lambda\mu - \nu), & \nabla\{2,2\} = 1 - \lambda^2 + \mu^2 - \nu^2, & \nabla\{2,3\} = 2(\mu\nu + \lambda) \\ \nabla\{3,1\} = 2(\nu\lambda + \mu), & \nabla\{3,2\} = 2(\mu\nu - \lambda), & \nabla\{3,3\} = 1 - \lambda^2 - \mu^2 + \nu^2 \end{array}$$

which will consequently express the values of the nine direction-cosines in the transformation from one set of rectangular co-ordinates to another, the formulæ of transformation being,

$$\begin{array}{ll} x = \{1,1\}\xi + \{1,2\}\eta + \{1,3\}\zeta & \xi = \{1,1\}x + \{2,1\}y + \{3,1\}z \\ y = \{2,1\}\xi + \{2,2\}\eta + \{2,3\}\zeta & \eta = \{1,2\}x + \{2,2\}y + \{3,2\}z \\ z = \{3,1\}\xi + \{3,2\}\eta + \{3,3\}\zeta & \zeta = \{1,3\}x + \{2,3\}y + \{3,3\}z \end{array}$$

A skew determinant is said to be *symmetrical* when

$$\begin{array}{llll} (1,1) = 0, & (1,2) + (2,1) = 0, & \cdot\cdot & (1,n) + (n,1) = 0 \\ (2,1) + (1,2) = 0, & (2,2) = 0, & \cdot\cdot & (2,n) + (n,2) = 0 \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (n,1) + (1,n) = 0, & (n,2) + (2,n) = 0, & \cdot\cdot & (n,n) = 0 \end{array}$$

The given and derived systems then give

$$u_1 + v_1 = 0, \quad u_2 + v_2 = 0, \cdot\cdot\ u_n + v_n = 0$$
$$u_1 + u_2 + \cdot\cdot + u_n = 0$$
$$v_1 + v_2 + \cdot\cdot + v_n = 0$$

and consequently,

$$\begin{array}{l} \nabla x_1 = [1,1]u_1 + [1,2]u_2 + \cdot\cdot + [1,n]u_n = [1,1]v_1 + [2,1]v_2 + \cdot\cdot + [n,1]v_n \\ \nabla x_2 = [2,1]u_1 + [2,2]u_2 + \cdot\cdot + [2,n]u_n = [1,2]v_1 + [2,2]v_2 + \cdot\cdot + [n,2]v_n \\ \cdot \quad \cdot \quad \cdot \quad \cdot\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\cdot \quad \cdot \\ \nabla x_n = [n,1]u_1 + [n,2]u_2 + \cdot\cdot + [n,n]u_n = [1,n]v_1 + [2,n]v_2 + \cdot\cdot + [n,n]v_n \end{array}$$

and consequently,

$$\begin{array}{llll}
2\nabla x_1 = \quad 0 & +([1,2] \;-[2,1])u_2 + & \cdot\cdot \; + ([1,n]-[n,1])u_n \\
2\nabla x_2 = ([2,1]-[1,2])u_1 + & \quad 0 & + \;\cdot\cdot\; + ([2,n]-[n,2])u_n \\
\quad\cdot & \quad\cdot & \quad\cdot\cdot & \quad\cdot \\
2\nabla x_n = ([n,1]-[1,n]u_1 & +([n,2] \;-[2,n])u_2 + & \cdot\cdot \; + \quad 0
\end{array}$$

on the other hand

$$\begin{array}{l}
0 = 2[1,1]u_1 + ([1,2] + [2,1])u_2 + \cdot\cdot + ([1,n] + [n,1])u_n \\
0 = ([2,1] + [1,2])u_1 + 2[2,2]u_2 + \cdot\cdot + ([2,n] + [n,2])u_n \\
\cdot \qquad \cdot \qquad \cdot \qquad \cdot\cdot \qquad \cdot \\
0 = ([n,1] + [1,n])u_1 + ([n,2] + [2,n])u_2 + \cdot\cdot + 2[n,n]u_n
\end{array}$$

and the comparison of these three systems gives either

$$\nabla = 0$$

$$\begin{array}{cccc}
* & [1,2], = [2,1], & \cdot\cdot & [1,n] = [n,1] \\
[2,1] = [1,2], & * & \cdot\cdot & [2,n] = [n,2] \\
\cdot & \cdot & \cdot\cdot & \cdot \\
[n,1] = [1,n], & \cdot & \cdot\cdot & *
\end{array}$$

or

$$\begin{array}{llll}
[1,1] = 0, & [1,2] + [2,1] = 0, & \cdot\cdot & [1,n] + [n,1] = 0 \\
[2,1] + [1,2] = 0, & [2,2] = 0, & \cdot\cdot & [2,n] + [n,2] = 0 \\
\quad\cdot & \quad\cdot & \cdot\cdot & \quad\cdot \\
[n,1] + [1,n] = 0, & [n,2] + [2,n] = 0, & \cdot\cdot & [n,n] = 0
\end{array}$$

and consequently either a symmetrical skew determinant of an even order, or a determinant of an odd order, always vanishes; but since it is found on trial that for $n = 1, 3, \ldots$, ∇ vanishes, while for $n = 2, 4, \ldots$, it does not, the following theorems may be enunciated.

THEOREM XIV. *A symmetrical skew determinant of an odd order in general vanishes, and the system has for its inverse an unsymmetrical skew system.*

THEOREM XV. *A symmetrical skew determinant of an even order does not in general vanish, but the system has for its inverse a symmetrical skew system.*

If n be even, a determinant of this class admits of the following reduction; it is easily shown that

$$\left|\begin{array}{cccc}
* & (1,2) & \cdot\cdot & (1,n) \\
(2,1) & * & \cdot\cdot & (2,n) \\
\cdot & \cdot & \cdot\cdot & \cdot \\
(n,1) & (n,2) & \cdot\cdot & *
\end{array}\right| = (1,2)^2 \left|\begin{array}{cccc}
* & (3,4) & \cdot\cdot & (3,n) \\
(4,3) & * & \cdot\cdot & (4,n) \\
\cdot & \cdot & \cdot\cdot & \cdot \\
(n,3) & (n,4) & \cdot\cdot & *
\end{array}\right| + 2(1,2)(1,3) \left|\begin{array}{cccc}
(3,4) & (3,5) & \cdot\cdot & (3,2) \\
* & (4,5) & \cdot\cdot & (4,2) \\
\cdot & \cdot & \cdot\cdot & \cdot \\
(n,4) & (n,5) & \cdot\cdot & (n,2)
\end{array}\right|$$

but

$$
\begin{aligned}
&\begin{vmatrix} (3,4) & (3,5) & \cdot\cdot & (3,2) \\ \ast & (4,5) & \cdot\cdot & (4,2) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (n,4) & (n,5) & \cdot\cdot & (n,2) \end{vmatrix}^2 = \begin{vmatrix} (3,2) & (3,4) & \cdot\cdot & (3,n) \\ (4,2) & \ast & \cdot\cdot & (4,n) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (n,2) & (n,4) & \cdot\cdot & \ast \end{vmatrix}^2 = \begin{vmatrix} (2,3) & (2,4) & \cdot\cdot & (2,n) \\ (4,3) & \ast & \cdot\cdot & (4,n) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (n,3) & (n,4) & \cdot\cdot & \ast \end{vmatrix} \\
&\times \begin{vmatrix} (3,2) & (3,4) & \cdot\cdot & (3,n) \\ (4,2) & \ast & \cdot\cdot & (4,n) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (n,2) & (n,4) & \cdot\cdot & \ast \end{vmatrix} = \begin{vmatrix} \ast & (2,4) & \cdot\cdot & (2,n) \\ (4,2) & \ast & \cdot\cdot & (4,n) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (n,2) & (n,4) & \cdot\cdot & \ast \end{vmatrix} \begin{vmatrix} \ast & (3,4) & \cdot\cdot & (3,n) \\ (4,3) & \ast & \cdot\cdot & (4,n) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (n,3) & (n,4) & \cdot\cdot & \ast \end{vmatrix}
\end{aligned}
$$

since the coefficients of (2,3) and (3,2), being symmetrical skew determinants of an odd order, vanish; so that finally,

$$
\begin{vmatrix} \ast & (1,2) & \cdot\cdot & (1,n) \\ (2,1) & \ast & \cdot\cdot & (2,n) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (n,1) & (n,2) & \cdot\cdot & \ast \end{vmatrix}^{\frac{1}{2}} = (1,2) \begin{vmatrix} \ast & (3,4) & \cdot\cdot & (3,n) \\ (4,3) & \ast & \cdot\cdot & (4,n) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (n,3) & (n,4) & \cdot\cdot & \ast \end{vmatrix}^{\frac{1}{2}} + (1,3) \begin{vmatrix} \ast & (4,5) & \cdot\cdot & (4,2) \\ (5,4) & \ast & \cdot\cdot & (5,2) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (2,4) & (2,5) & \cdot\cdot & \ast \end{vmatrix}^{\frac{1}{2}} + \cdot\cdot
$$

If in the determinant

$$
\begin{vmatrix} (1,1) & (1,2) & \cdot\cdot & (1,n) \\ -(1,2) & (2,2) & \cdot\cdot & (2,n) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ -(1,n) & -(2,n) & \cdot\cdot & (n,n) \end{vmatrix}
$$

the quantities (1,1), (2,2), .. (n,n) be put simultaneously equal to zero, the terms independent of these quantities will remain; if all but one of them be put equal to zero, those terms which involve that quantity will remain; if all but two be put equal to zero, those terms which involve their product will remain, and so on; so that a general skew determinant may be thus expressed;

$$
\begin{aligned}
&\begin{vmatrix} (1,1) & (1,2) & \cdot\cdot & (1,n) \\ -(1,2) & (2,2) & \cdot\cdot & (2,n) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ -(1,n) & -(2,n) & \cdot\cdot & (n,n) \end{vmatrix} = \begin{vmatrix} \ast & (1,2) & \cdot\cdot & (1,n) \\ -(1,2) & \ast & \cdot\cdot & (2,n) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ -(1,n) & -(2,n) & \cdot\cdot & \ast \end{vmatrix} \\
&+(1,1) \begin{vmatrix} \ast & (2,3) & \cdot\cdot & (2,n) \\ -(2,3) & \ast & \cdot\cdot & (3,n) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ -(2,n) & -(3,n) & \cdot\cdot & \ast \end{vmatrix} + (2,2) \begin{vmatrix} \ast & (3,4) & \cdot\cdot & (3,1) \\ -(3,4) & \ast & \cdot\cdot & (4,1) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ -(3,1) & -(4,1) & \cdot\cdot & \ast \end{vmatrix} \\
&+ \cdot\cdot + (1,1)\,(2,2) \begin{vmatrix} \ast & (3,4) & \cdot\cdot & (3,n) \\ -(3,4) & \ast & \cdot\cdot & (4,n) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ -(3,n) & -(4,n) & \cdot\cdot & \ast \end{vmatrix} + \cdot\cdot
\end{aligned}
$$

Hence if n be even,

$$\begin{vmatrix} (1,1) & (1,2)\cdot\cdot(1,n) \\ -(1,2) & (2,2)\cdot\cdot(2,n) \\ \cdot & \cdot\ \cdot\cdot\ \cdot \\ -(1,n) & -(2,n)\cdot\cdot(n,n) \end{vmatrix} = \left((1,2) \begin{vmatrix} \ast & (3,4)\cdot\cdot(3,n) \\ -(3,4) & \ast\ \cdot\cdot(4,n) \\ \cdot & \cdot\ \cdot\cdot\ \cdot \\ -(3,n) & -(4,n)\cdot\cdot\ \ast \end{vmatrix}^{\frac{1}{2}} + (1,3) \begin{vmatrix} \ast & (4,5)\cdot\cdot(4,2) \\ -(4,5) & \ast\ \cdot\cdot(5,2) \\ \cdot & \cdot\ \cdot\cdot\ \cdot \\ -(4,2) & -(5,2)\cdot\cdot\ \ast \end{vmatrix}^{\frac{1}{2}} + \cdot\cdot\right)^2$$

$$+ (1,1)(2,2)\left((3,4) \begin{vmatrix} \ast & (5,6)\cdot\cdot(5,n) \\ -(5,6) & \ast\ \cdot\cdot(6,n) \\ \cdot & \cdot\ \cdot\cdot\ \cdot \\ -(5,n) & -(6,n)\cdot\cdot\ \ast \end{vmatrix}^{\frac{1}{2}} + (3,5) \begin{vmatrix} \ast & (6,7)\cdot\cdot(6,4) \\ -(6,7) & \ast\ \cdot\cdot(7,4) \\ \cdot & \cdot\ \cdot\cdot\ \cdot \\ -(6,4) & -(7,4)\cdot\cdot\ \ast \end{vmatrix}^{\frac{1}{2}} + \cdot\cdot\right)^2$$

$$+ \cdot\cdot + (1,1)(2,2)\cdot\cdot(n,n)$$

and if n be odd,

$$\begin{vmatrix} (1,1) & (1,2)\cdot\cdot(1,n) \\ -(1,2) & (2,2)\cdot\cdot(2,n) \\ \cdot & \cdot\ \cdot\cdot\ \cdot \\ -(1,n) & -(2,n)\cdot\cdot(n,n) \end{vmatrix} = (1,1)\left((2,3) \begin{vmatrix} \ast & (4,5)\cdot\cdot(4,n) \\ -(4,5) & \ast\ \cdot\cdot(5,n) \\ \cdot & \cdot\ \cdot\cdot\ \cdot \\ -(4,n) & -(5,n)\cdot\cdot\ \ast \end{vmatrix} + (2,4) \begin{vmatrix} \ast & (5,6)\cdot\cdot(5,3) \\ -(5,6) & \ast\ \cdot\cdot(6,3) \\ \cdot & \cdot\ \cdot\cdot\ \cdot \\ -(5,3) & -(6,3)\cdot\cdot\ \ast \end{vmatrix} + \cdot\cdot\right)^2$$

$$+ (2,2)\left((3,4) \begin{vmatrix} \ast & (5,6)\cdot\cdot(5,1) \\ -(5,6) & \ast\ \cdot\cdot(6,1) \\ \cdot & \cdot\ \cdot\cdot\ \cdot \\ -(5,1) & -(6,1)\cdot\cdot\ \ast \end{vmatrix} + (3,5) \begin{vmatrix} \ast & (6,7)\cdot\cdot(6,4) \\ -(6,7) & \ast\ \cdot\cdot(7,4) \\ \cdot & \cdot\ \cdot\cdot\ \cdot \\ -(6,4) & -(7,4)\cdot\cdot\ \ast \end{vmatrix} + \cdot\cdot\right)^2$$

$$+ \cdot\cdot + (1,1)(2,2)\cdot\cdot(n,n)$$

§ X. *On Functional Determinants.*

THERE is a class of determinants whose constituents are differential coefficients of functions of variables, and which are called *functional determinants;* they are capable of numerous applications, and although subject to the same general laws of combination and development as ordinary determinants, possess many peculiarities, which make it necessary to discuss them separately.

If $f, f_1, \ldots f_n$ be functions of $x, x_1, \ldots x_n$ not independent of one another, but connected by some equation, such as,

$$\Pi = 0$$

then the equations

$$\frac{d\Pi}{dx} = 0, \quad \frac{d\Pi}{dx_1} = 0, \cdot\cdot \frac{d\Pi}{dx_n} = 0,$$

hold good on account of the independence of x, x_1, .. x_n; hence

$$\begin{gathered}
\frac{d\Pi}{df}\,\frac{df}{dx} + \frac{d\Pi}{df_1}\,\frac{df_1}{dx} + \cdot\cdot + \frac{d\Pi}{df_n}\,\frac{df_n}{dx} = 0 \\
\frac{d\Pi}{df}\,\frac{df}{dx_1} + \frac{d\Pi}{df_1}\,\frac{df_1}{dx_1} + \cdot\cdot + \frac{d\Pi}{df_n}\,\frac{df_n}{dx_1} = 0 \\
\cdot \qquad \cdot \qquad \cdot\cdot \qquad \cdot \\
\frac{d\Pi}{df}\,\frac{df}{dx_n} + \frac{d\Pi}{df_1}\,\frac{df_1}{dx_1} + \cdot\cdot + \frac{d\Pi}{df_n}\,\frac{df_n}{dx_n} = 0
\end{gathered}$$

and consequently

$$\begin{vmatrix}
\frac{df}{dx} & \frac{df_1}{dx} & \cdot\cdot & \frac{df_n}{dx} \\
\frac{df}{dx_1} & \frac{df_1}{dx_1} & \cdot\cdot & \frac{df_n}{dx_1} \\
\cdot & \cdot & \cdot\cdot & \cdot \\
\frac{df}{dx_n} & \frac{df_1}{dx_n} & \cdot\cdot & \frac{df_n}{dx_n}
\end{vmatrix} = 0$$

Conversely, if the latter equation hold good, the preceding system may always be satisfied, and the functions f, f_1, .. f_n are not independent.

If f, f_1, .. f_n be functions of the variables x, x_1, .. x_n, independent of one another, and there be given the system of equations

$$\begin{gathered}
s\frac{df}{dx} + s_1\frac{df}{dx_1} + \cdot\cdot + s_n\frac{df}{dx_n} = t \\
s\frac{df_1}{dx} + s_1\frac{df_1}{dx_1} + \cdot\cdot + s_n\frac{df_1}{dx_n} = t_1 \\
\cdot \qquad \cdot \qquad \cdot\cdot \qquad \cdot \\
s\frac{df_n}{dx} + s_1\frac{df_n}{dx_1} + \cdot\cdot + s_n\frac{df_n}{dx_n} = t_n
\end{gathered}$$

then, considering x, x_1, .. x_n as functions of f, f_1, .. f_n, the equations

$$\begin{gathered}
\frac{dx}{dx} = 1, \quad \frac{dx}{dx} = 0, \quad \cdot\cdot \quad \frac{dx}{dx_n} = 0 \\
\frac{dx_1}{dx} = 0, \quad \frac{dx_1}{dx_1} = 1, \quad \cdot\cdot \quad \frac{dx_1}{dx_n} = 0 \\
\cdot \qquad \cdot \qquad \cdot\cdot \qquad \vdots \\
\frac{dx_n}{dx} = 0, \quad \frac{dx_n}{dx_1} = 0, \quad \cdot\cdot \quad \frac{dx_n}{dx_n} = 1
\end{gathered}$$

give

$$\begin{array}{lll}
\frac{df}{dx}\frac{dx}{df}+\frac{df_1}{dx}\frac{dx}{df_1}+\cdot\cdot=1, & \frac{df}{dx_1}\frac{dx}{df}+\frac{df_1}{dx_1}\frac{dx}{df_1}+\cdot\cdot=0, & \cdot\cdot\ \frac{df}{dx_n}\frac{dx}{df}+\frac{df_1}{dx_n}\frac{dx}{df_1}=0 \\
\frac{df}{dx}\frac{dx_1}{df}+\frac{df_1}{dx}\frac{dx_1}{df_1}+\cdot\cdot=0, & \frac{df}{dx_1}\frac{dx_1}{df}+\frac{df_1}{dx_1}\frac{dx_1}{df_1}+\cdot\cdot=1, & \cdot\cdot\ \frac{df}{dx_n}\frac{dx_1}{df}+\frac{df_1}{dx_n}\frac{dx_1}{df_1}+\cdot\cdot=0 \\
\cdot\quad\cdot\quad\cdot\cdot & \cdot\quad\cdot\quad\cdot\cdot & \cdot\quad\cdot \\
\frac{df}{dx}\frac{dx_n}{df}+\frac{df_1}{dx}\frac{dx_n}{df_1}+\cdot\cdot=0, & \frac{df}{dx_1}\frac{dx_n}{df}+\frac{df_1}{dx_1}\frac{dx_n}{df_1}+\cdot\cdot=0, & \cdot\ \frac{df}{dx_n}\frac{dx_n}{df}+\frac{df_1}{dx_n}\frac{dx_n}{df_1}+\cdot\cdot=1
\end{array}$$

and consequently

$$\begin{array}{l}
t\frac{dx}{df}+t_1\frac{dx}{df_1}+\cdot\cdot+t_n\frac{dx}{df_n}=s \\
t\frac{dx_1}{df}+t_1\frac{dx_1}{df_1}+\cdot\cdot+t_n\frac{dx_1}{df_n}=s_1 \\
\cdot\quad\cdot\quad\cdot\cdot\quad\cdot\quad\cdot \\
t\frac{dx_1}{df}+t_1\frac{dx_1}{df_1}+\cdot\cdot+t_n\frac{dx_1}{df_n}=s_n
\end{array}$$

If $f, f_1, \cdot\cdot f_n$ be functions of the variables $x, x_1, \cdot\cdot x_n$, independent of one another, and there be given the equations

$$\begin{array}{l}
u\frac{df}{dx}+u_1\frac{df_1}{dx}+\cdot\cdot+u_n\frac{df_n}{dx}=v \\
u\frac{df}{dx_1}+u_1\frac{df_1}{dx_1}+\cdot\cdot+u_n\frac{df_n}{dx_1}=v_1 \\
\cdot\quad\cdot\quad\cdot\cdot\quad\cdot\quad\cdot \\
u\frac{df}{dx_n}+u_1\frac{df_1}{dx_n}+\cdot\cdot+u_n\frac{df_n}{dx_n}=v_n
\end{array}$$

Then since

$$\begin{array}{lll}
\frac{df}{dx}\frac{dx}{df}+\frac{df}{dx_1}\frac{dx_1}{df}+\cdot\cdot=1, & \frac{df_1}{dx}\frac{dx}{df}+\frac{df_1}{dx_1}\frac{dx_1}{df}+\cdot\cdot=0, & \cdot\cdot\ \frac{df_n}{dx}\frac{dx}{df}+\frac{df_n}{dx_1}\frac{dx_1}{df}+\cdot\cdot=0 \\
\frac{df}{dx}\frac{dx}{df_1}+\frac{df}{dx_1}\frac{dx_1}{df_1}+\cdot\cdot=0, & \frac{df_1}{dx}\frac{dx}{df_1}+\frac{df_1}{dx_1}\frac{dx_1}{df_1}+\cdot\cdot=1, & \cdot\cdot\ \frac{df_n}{dx}\frac{dx}{df_1}+\frac{df_n}{dx_1}\frac{dx_1}{df_1}+\cdot\cdot=0 \\
\cdot\quad\cdot\quad\cdot\cdot & \cdot\quad\cdot\quad\cdot\cdot & \cdot\cdot\quad\cdot\quad\cdot\quad\cdot\cdot \\
\frac{df}{dx}\frac{dx}{df_n}+\frac{df}{dx_1}\frac{dx_1}{df_n}+\cdot\cdot=0, & \frac{df_1}{dx}\frac{dx}{df_n}+\frac{df_1}{dx_1}\frac{dx_1}{df_n}+\cdot\cdot=0, & \cdot\cdot\ \frac{df_n}{dx}\frac{dx}{df_n}+\frac{df_n}{dx_1}\frac{dx_1}{df_n}+\cdot\cdot=1
\end{array}$$

there may be deduced

$$\begin{array}{l}
v\frac{dx}{df}+v_1\frac{dx_1}{df}+\cdot\cdot+v_n\frac{dx_n}{df}=u \\
v\frac{dx}{df_1}+v_1\frac{dx_1}{df_1}+\cdot\cdot+v_n\frac{dx_n}{df_1}=u_1 \\
\cdot\quad\cdot\cdot\quad\cdot\quad\cdot \\
v\frac{dx}{df_n}+v_1\frac{dx_1}{df_n}+\cdot\cdot+v_n\frac{dx_n}{df_n}=u_n
\end{array}$$

Hence also by what has been proved above, $f, f_1, \cdot\cdot f_n$ being independent, if

$$t = 0, \quad t_1 = 0, \quad \cdot\cdot \quad t_n = 0$$

then also

$$s = 0, \quad s_1 = 0, \quad \cdot\cdot \quad s_n = 0$$

and conversely.

Comparing the solutions of the systems of equations given above with those which would be found by the ordinary method of determinants, and writing

$$\nabla = \begin{vmatrix} \frac{df}{dx} & \frac{df}{dx_1} & \cdot\cdot & \frac{df}{dx_n} \\ \frac{df_1}{dx} & \frac{df_1}{dx_1} & \cdot\cdot & \frac{df_1}{dx_n} \\ \cdot & \cdot & \cdot\cdot & \cdot \\ \frac{df_n}{dx} & \frac{df_n}{dx_1} & \cdot\cdot & \frac{df_n}{dx_n} \end{vmatrix}$$

it is seen that

$$\begin{array}{lll} \nabla\frac{dx}{df} = [0,0], & \nabla\frac{dx_1}{df} = [0,1], \cdot\cdot & \nabla\frac{dx_n}{df} = [0,n] \\ \nabla\frac{dx}{df_1} = [1,0], & \nabla\frac{dx_1}{df_1} = [1,1], \cdot\cdot & \nabla\frac{dx_n}{df_1} = [1,n] \\ \cdot & \cdot \quad \cdot & \cdot\cdot \quad \cdot \\ \nabla\frac{dx}{df_n} = [n,0], & \nabla\frac{dx_1}{df_n} = [n,1], \cdot\cdot & \nabla\frac{dx_n}{df_n} = [n,n] \end{array}$$

which may also be written in the following forms,

$$\begin{array}{lll} \frac{d\nabla}{d\frac{df}{dx}} = \nabla\frac{dx}{df}, & \frac{d\nabla}{d\frac{df}{dx_1}} = \nabla\frac{dx_1}{df}, \cdot\cdot & \frac{d\nabla}{d\frac{df}{dx_n}} = \nabla\frac{dx_n}{df} \\ \frac{d\nabla}{d\frac{df_1}{dx}} = \nabla\frac{dx}{df_1}, & \frac{d\nabla}{d\frac{df_1}{dx_1}} = \nabla\frac{dx_1}{df_1}, \cdot\cdot & \frac{d\nabla}{d\frac{df_1}{dx_n}} = \nabla\frac{dx_n}{df_1} \\ \cdot \quad \cdot & \cdot \quad \cdot & \cdot\cdot \quad \cdot \quad \cdot \\ \frac{d\nabla}{d\frac{df_n}{dx}} = \nabla\frac{dx}{df_n}, & \frac{d\nabla}{d\frac{df_n}{dx_1}} = \nabla\frac{dx_1}{df_n}, \cdot\cdot & \frac{d\nabla}{d\frac{df_n}{dx_n}} = \nabla\frac{dx_n}{df_n} \end{array}$$

and inversely,

$$\begin{array}{llll}
\dfrac{d\nabla}{d\dfrac{dx}{df}} = -\nabla\dfrac{df}{dx}, & \dfrac{d\nabla}{d\dfrac{dx}{df_1}} = -\nabla\dfrac{df_1}{dx}, & \cdot\cdot & \dfrac{d\nabla}{d\dfrac{dx}{df}} = -\nabla\dfrac{df_n}{dx} \\
\dfrac{d\nabla}{d\dfrac{dx_1}{df}} = -\nabla\dfrac{df}{dx_1}, & \dfrac{d\nabla}{d\dfrac{dx}{df_1}} = -\nabla\dfrac{df_1}{dx_1}, & \cdot\cdot & \dfrac{d\nabla}{d\dfrac{dx_1}{df_n}} = -\nabla\dfrac{df_n}{dx_1} \\
\cdot \quad \cdot & \cdot \quad \cdot & \cdot\cdot & \cdot \quad \cdot \\
\dfrac{d\nabla}{d\dfrac{dx_n}{df}} = -\nabla\dfrac{df}{dx_n}, & \dfrac{d\nabla}{d\dfrac{dx_n}{df_1}} = -\nabla\dfrac{df_1}{dx_n}, & \cdot\cdot & \dfrac{d\nabla}{d\dfrac{dx_n}{df_n}} = -\nabla\dfrac{df_n}{dx_n}
\end{array}$$

If the determinant formed from the expressions on the left-hand sides of the system in the preceding page be equated to that formed by the corresponding expressions on the right-hand side, the theorem for the multiplication of determinants gives at once

$$\begin{vmatrix} \dfrac{df}{dx} & \dfrac{df}{dx_1} & \cdot\cdot & \dfrac{df}{dx_n} \\ \dfrac{df_1}{dx} & \dfrac{df_1}{dx_1} & \cdot\cdot & \dfrac{df_1}{dx_n} \\ \cdot & \cdot & \cdot\cdot & \cdot \\ \dfrac{df_n}{dx} & \dfrac{df_n}{dx_1} & \cdot\cdot & \dfrac{df_n}{dx_n} \end{vmatrix} \begin{vmatrix} \dfrac{dx}{df} & \dfrac{dx}{df_1} & \cdot\cdot & \dfrac{dx}{df_n} \\ \dfrac{dx_1}{df} & \dfrac{dx_1}{df_1} & \cdot\cdot & \dfrac{dx_1}{df_n} \\ \cdot & \cdot & \cdot\cdot & \cdot \\ \dfrac{dx_n}{df} & \dfrac{dx_n}{df_1} & \cdot\cdot & \dfrac{dx_n}{df_n} \end{vmatrix} = 1$$

the same result might be obtained by substituting for [0,0], [0,1], .. in the equation (see § VI.)

$$\begin{vmatrix} [0,0] & [0,1] & \cdot\cdot & [0,n] \\ [1,0] & [1,1] & \cdot\cdot & [1,n] \\ \cdot & \cdot & \cdot\cdot & \cdot \\ [n,0] & [n,1] & \cdot\cdot & [n,n] \end{vmatrix} = \nabla^n$$

In connexion with which the following formula may be noticed:

If $f, f_1, \dots f_n$ are related by the $(n+1)$ equations,

$$F = 0, \qquad F_1 = 0, \cdot\cdot \; F_n = 0$$

n of the $(n+1)$ variables may be eliminated from each of these last functions, so that each may be considered as a function of a single variable, and of the functions $f, f_1, \ldots f_n$; and consequently

$$\begin{array}{lll}
\frac{dF}{dx}+\frac{dF}{df}\frac{df}{dx}+\cdot\cdot=0, & \frac{dF}{dx_1}+\frac{dF}{df}\frac{df}{dx_1}+\cdot\cdot=0, \ldots & \frac{dF}{dx_n}+\frac{dF}{df}\frac{df}{dx_n}+\cdot\cdot=0 \\
\frac{dF_1}{dx}+\frac{dF_1}{df}\frac{df}{dx}+\cdot\cdot=0, & \frac{dF_1}{dx_1}+\frac{dF_1}{df}\frac{df}{dx_1}+\cdot\cdot=0, \ldots & \frac{dF_1}{dx_n}+\frac{dF_1}{df}\frac{df}{dx_n}+\cdot\cdot=0 \\
\cdot \quad \cdot & \cdot \quad \cdot & \cdot\cdot \quad \cdot \quad \cdot \quad \cdot \\
\frac{dF_n}{dx}+\frac{dF_n}{df}\frac{df}{dx}+\cdot\cdot=0, & \frac{dF_n}{dx_1}+\frac{dF_n}{df}\frac{df}{dx_1}+\cdot\cdot=0, \ldots & \frac{dF_n}{dx_n}+\frac{dF_n}{df}\frac{df}{dx_n}+\cdot\cdot=0
\end{array}$$

transposing the terms $\frac{dF}{dx}, \frac{dF}{dx_1}, \ldots$ and equating the determinant formed from the expressions on the left-hand sides of these equations to that formed from those on the right, the formula for the multiplication of determinants gives

$$(-)^{n+1}\begin{vmatrix}
\frac{dF}{dx} & \frac{dF}{dx_1} & \ldots & \frac{dF}{dx_n} \\
\frac{dF_1}{dx} & \frac{dF_1}{dx_1} & \ldots & \frac{dF_1}{dx_n} \\
\cdot & \cdot & \cdot\cdot & \cdot \\
\frac{dF_n}{dx} & \frac{dF_n}{dx_1} & \ldots & \frac{dF_n}{dx_n}
\end{vmatrix} = \begin{vmatrix}
\frac{dF}{df} & \frac{dF}{df_1} & \ldots & \frac{dF}{df_n} \\
\frac{dF_1}{df} & \frac{dF_1}{df_1} & \ldots & \frac{dF_1}{df_n} \\
\cdot & \cdot & \cdot\cdot & \cdot \\
\frac{dF_n}{df} & \frac{dF_n}{df_1} & \ldots & \frac{dF_n}{df_n}
\end{vmatrix} \begin{vmatrix}
\frac{df}{dx} & \frac{df}{dx_1} & \ldots & \frac{df}{dx_n} \\
\frac{df_1}{dx} & \frac{df_1}{dx_1} & \ldots & \frac{df_1}{dx_n} \\
\cdot & \cdot & \cdot\cdot & \cdot \\
\frac{df_n}{dx} & \frac{df_n}{dx_1} & \ldots & \frac{df_n}{dx_n}
\end{vmatrix}$$

Determinants are useful also in the transformation of multiple integrals, and lead to a simple solution of the general problem which may be thus stated; let

$$V = \int\int\cdot\cdot\, U\, dx\, dx_1 \cdot\cdot\, dx_n$$

and suppose it be required to transform the integral to one in which $y, y_1, \ldots y_n$ shall be the independent variables, then

$$\begin{aligned}
dx &= \frac{dx}{dy}dy + \frac{dx}{dy_1}dy_1 + \cdot\cdot + \frac{dx}{dy_n}dy_n \\
dx_1 &= \frac{dx_1}{dy}dy + \frac{dx_1}{dy_1}dy_1 + \cdot\cdot + \frac{dx_1}{dy_n}dy_n \\
&\cdot \quad \cdot \quad \cdot \quad \cdot\cdot \quad \cdot \\
dx_n &= \frac{dx_n}{dy}dy + \frac{dx_n}{dy_1}dy_1 + \cdot\cdot + \frac{dx_n}{dy_n}dy_n
\end{aligned}$$

and since $x, x_1, \ldots x_n$ vary independently, we must put

$$dx_1 = 0, \ dx_2 = 0, \cdot\cdot\ dx_n = 0$$

in order to find dx; this gives

$$\nabla\, dy = \nabla_1 dx$$

and consequently dx and dy vanish together, (the value of ∇_1 is obvious). And therefore for the determination of the remaining differentials there exist the equations

$$\begin{aligned}
dx_1 &= \frac{dx_1}{dy_1}dy_1 + \frac{dx_1}{dy_2}dy_2 + \cdot\cdot + \frac{dx_1}{dy_n}dy_n \\
dx_2 &= \frac{dx_2}{dy_1}dy_1 + \frac{dx_2}{dy_2}dy_2 + \cdot\cdot + \frac{dx_2}{dy_n}dy_n \\
\cdot \quad & \quad \cdot \qquad\quad \cdot \qquad\quad \cdot\cdot \qquad \cdot \\
dx_n &= \frac{dx_n}{dy_1}dy_1 + \frac{dx_n}{dy_2}dy_2 + \cdot\cdot + \frac{dx_n}{dy_n}dy_n
\end{aligned}$$

proceeding as before, and putting

$$dx_2 = 0, \quad dx_3 = 0, \cdot\cdot\ dx_n = 0$$

there will result

$$\nabla_1 dy_1 = \nabla_2 dx_1$$

and so on successively until

$$\nabla_n dy_n = dx_n$$

hence multiplying all these expressions together, and dividing out the common factor,

$$\nabla_1\ \nabla_2 \cdot\cdot\ \nabla_n$$

there will finally result

$$\nabla dy\ dy_1 \cdot\cdot\ dy_n = dx\ dx_1 \cdot\cdot\ dx_n$$

and consequently

$$V = \int\int \cdot\cdot\ U\, \nabla\, dy\ dy_1 \cdot\cdot\ dy_n$$

If

$$U = 0$$

and

$$(0,1) = \frac{dU}{dx_1} = 0, \qquad (0,2) = \frac{dU}{dx_2} = 0, \ \cdot\cdot\ \ (0,n) = \frac{dU}{dx_n} = 0$$

then differentiating,

$$\begin{aligned}
(1,1)\,dx_1 + (2,1)\,dx_2 + \cdot\cdot + (n,1)\,dx_n &= 0 \\
(1,2)\,dx_1 + (2,2)\,dx_2 + \cdot\cdot + (n,2)\,dx_n &= 0 \\
\cdot \qquad\quad \cdot \qquad\quad \cdot\cdot \qquad \cdot \qquad & \\
(1,n)\,dx_1 + (2,n)\,dx_2 + \cdot\cdot + (n,n)\,dx_n &= 0
\end{aligned}$$

and

$$\nabla = \begin{vmatrix} (1,1) & (1,2) & \cdot\cdot & (1,n) \\ (2,1) & (2,2) & \cdot\cdot & (2,n) \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (n,1) & (n,2) & \cdot\cdot & (n,n) \end{vmatrix} = 0$$

and the preceding system gives

$$\begin{array}{llll}\lambda_1 dx_1 = [1,1], & \lambda_1 dx_2 = [1,2], & \cdot\cdot & \lambda_1 dx_n = [1,n] \\ \lambda_2 dx_1 = [2,1], & \lambda_2 dx_2 = [2,2], & \cdot\cdot & \lambda_2 dx_n = [2,n] \\ \cdot & \cdot & \cdot\cdot & \cdot \\ \lambda_n dx_1 = [n,1], & \lambda_n dx_2 = [n,2], & \cdot\cdot & \lambda_n dx_n = [n,n]\end{array}$$

$\lambda_1, \lambda_2, \ldots \lambda_n$ being indeterminate multipliers. Suppose moreover that

$$\{0,1\} = \frac{d\nabla}{dx_1},\ \{0.2\} = \frac{d\nabla}{dx_2},\ \cdot\cdot\ \{0,n\} = \frac{d\nabla}{dx_n}$$

then, it being observed that $(i,j) = (j,i)$, and $[i,j] = [j,i]$, it follows that

$$\begin{aligned}\{0,1\} &= (1,1,1)\,[1,1] + (1,2,1)\,[1,2] + \cdot\cdot + (1,n,1)\,[1,n] \\ &\quad + (1,1,2)\,[2,1] + (1,2,2)\,[2,2] + \cdot\cdot + (1,n,2)\,[2,n] \\ &\quad + \quad \cdot \quad\quad \cdot \quad\quad \cdot\cdot \quad\quad \cdot \\ &\quad + (1,1,n)\,[n,1] + (1,2,n)\,[n,2] + \cdot\cdot + (1,n,n)\,[n,n] \\ &= \lambda_1\{(1,1,1)\,dx_1 + (1,1,2)\,dx_2 + \cdot\cdot + (1,1,n)\,dx_n\} \\ &\quad + \lambda_2\{(1,2,1)\,dx_1 + (1,2,2)\,dx_2 + \cdot\cdot + (1,n,n)\,dx_n\} \\ &\quad + \quad \cdot \quad\quad \cdot \quad\quad \cdot\cdot \quad\quad \cdot \\ &\quad + \lambda_n\{(1,n,1)\,dx_1 + (1,n,2)\,dx_2 + \cdot\cdot + (1,n,n)\,dx_n\}\end{aligned}$$

whence the following system may be formed

$$\begin{aligned}\{0,1\} &= \lambda_1 D(1,1) + \lambda_2 D(1,2) + \cdot\cdot + \lambda_n(1,n) \\ \{0,2\} &= \lambda_1 D(2,1) + \lambda_2 D(2,2) + \cdot\cdot + \lambda_n(2,n) \\ \cdot \quad &\quad \cdot \quad\quad \cdot \quad\quad \cdot\cdot \quad\quad \cdot \\ \{0,n\} &= \lambda_1 D(n,1) + \lambda_2 D(n,2) + \cdot\cdot + \lambda_n D(n,n)\end{aligned}$$

where D indicates the total differential; but since

$$\lambda_2 dx_1 = \lambda_1 dx_2,\ \lambda_3 dx_1 = \lambda_1 dx_3,\ \cdot\cdot\ \lambda_n dx_1 = \lambda_1 dx_n$$

consequently

$$\lambda_1 : \lambda_2 : \cdot\cdot\ \lambda_n = dx_1 : dx_2 : \cdot\cdot\ dx_n$$

and if θ be the common ratio of each quantity on the left-hand side of this system to the corresponding quantity on the right-hand side, the above system may be written thus,

$$\{0,1\} = \theta D^2(0,1),\ \{0,2\} = \theta D^2(0,2),\ \cdot\cdot\ \{0,n\} = \theta D^2(0,n)$$

but since

$$(0,1) = 0,\ (0,2) = 0, = (0,n) = 0$$

therefore also

$$D^2(0,1) = 0, \qquad D^2(0,2) = 0,\ \cdot\cdot\ D^2(0,n) = 0$$

and consequently

$$\{0,1\} = 0, \qquad \{0,2\} = 0, \;\cdot\cdot\; \{0,n\} = 0$$

these again give

$$\begin{array}{l}
\{1,1\}dx_1 + \{1,2\}dx_2 + \cdot\cdot + \{1,n\}dx_n = 0 \\
\{2,1\}dx_1 + \{2,2\}dx_2 + \cdot\cdot + \{2,n\}dx_n = 0 \\
\quad\cdot \qquad\quad \cdot \qquad \cdot\cdot \qquad \cdot \\
\{n,1\}dx_1 + \{n,2\}dx_2 + \cdot\cdot + \{n,n\}dx_n = 0
\end{array}$$

and consequently

$$\begin{vmatrix}
\{1,1\}, & \{1,2\}, & \cdot\cdot & \{1,n\} \\
\{2,1\}, & \{2,2\}, & \cdot\cdot & \{2,n\} \\
\cdot & \cdot & \cdot\cdot & \cdot \\
\{n,1\}, & \{n,2\}, & \cdot\cdot & \{n,n\}
\end{vmatrix} = 0$$

From the above formulæ the following relations are easily deduced,

$$\begin{array}{llll}
[1,1] : [1,2] : \cdot\cdot[1,n] = & [\{1,1\}] : [\{1,2\}] : \cdot\cdot[\{1,n\}] \\
[2,1] : [2,2] : \cdot\cdot[2,n] & : [\{2,1\}] : [\{2,2\}] : \cdot\cdot[\{2,n\}] \\
\cdot \quad \cdot \quad \cdot\cdot \quad \cdot & \quad \cdot \qquad \cdot \qquad \cdot\cdot \quad \cdot \\
[n,1] : [2,n] : \cdot\cdot[n,n] & : [\{n,1\}] : [\{n,2\}] : \cdot\cdot[\{n,n\}]
\end{array}$$

where $[\{i,j\}]$ is the inverse of $\{j,i\}$. This system also involves the following,

$$\begin{array}{ll}
(1,1) : (1,2) : \cdot\cdot(1,n) = & \{1,1\} : \{1,2\} : \cdot\cdot\{1,n\} \\
: (2,1) : (2,2) : \cdot\cdot(2,n) & : \{2,1\} : \{2,2\} : \cdot\cdot\{2,n\} \\
\cdot \quad \cdot \quad \cdot\cdot \quad \cdot & \quad \cdot \qquad \cdot \qquad \cdot\cdot \quad \cdot \\
: (n,1) : (n,2) : \cdot\cdot(n,n) & : \{n,1\} : \{n,2\} : \cdot\cdot\{n,n\}
\end{array}$$

This last system of relations was given by Dr. Hesse (Crelle, tom. XL.) with a demonstration by Jacobi.

The above equations are applicable to certain questions in Geometry. Thus, in the case where $n = 3$, the equation

$$\mathrm{U} = 0$$

represents a cone when x, x_1, x_2 are the co-ordinates of a point; and it represents a plane curve when the ratios of any two variables to the third are the co-ordinates of a point in the plane. This in fact is the same thing as forming a plane curve by the intersection of a cone and a plane. In order to find the condition for a point of inflexion on the plane curve, it will therefore be sufficient to find the condition that the principal (and consequently all the) radii of curvature of the cone at a certain point (x, x_1, x_2) shall be infinite. The condition in question is, as is well known, represented by the system

$$(0,1) = 0, \qquad (0,2) = 0, \qquad (0,3) = 0$$

or

$$\begin{aligned}(1,1)dx_1 + (1,2)dx_2 + (1,3)dx_3 &= 0\\(2,1)dx_1 + (2,2)dx_2 + (2,3)dx_3 &= 0\\(3,1)dx_1 + (3,2)dx_2 + (3,3)dx_3 &= 0\end{aligned}$$

and consequently

$$\begin{vmatrix}(1,1) & (1,2) & (1,3)\\(2,1) & (2,2) & (2,3)\\(3,1) & (3,2) & (3,3)\end{vmatrix} = 0 .$$

This equation combined with that to the curve will determine the points of inflexion of the curve. It follows therefore that a curve of the nth order has in general $3n\,(n-2)$ points of inflexion.

The following example is taken from the Cambridge and Dublin Mathematical Journal.

" Jacobi, in a very elaborate memoir, 'Theoria novi multiplicatoris systemati æquationum differentialium vulgarium applicandi,' has demonstrated a remarkable property of an extensive class of differential equations, namely, that when all the integrals of the system except a single one are known, the remaining integral can always be determined by a quadrature. Included in the class in question are, as Jacobi proceeds to show, the differential equations corresponding to any dynamical problem in which neither the forces nor the equations of condition involve the velocities; i. e. in all ordinary dynamical problems when all the integrals but one are known the remaining integral can be determined by quadratures. In the case where the forces and equations of condition are likewise independent of the time, it is immediately seen that the system may be transformed into a system in which the number of equations is less by unity than in the original one, and which does not involve the time, which may afterwards be determined by a quadrature, and Jacobi's theorem applying to this new system, he arrives at the proposition, ' In any dynamical problem where the forces and equations of condition contain only the coordinates of the different points of the system, when all the integrals but two are determined, the remaining integrals may be found by quadratures only.' In the following paper, which contains the demonstrations of these propositions, the analysis employed by Jacobi has been considerably varied in the details, but the leading features of it are preserved.

" § 1. Let the variables $x, y, z, \ldots$ &c. be connected with the variables $u, v, w, \ldots$ by the same number of equations, so that the variables of each set may be considered as functions of those of the other set. And assume

$$dx\, dy \cdots = \nabla\, du\, dv \cdots$$

If from the functions which equated to zero express the relations between the two sets of variables we form two determinants, the former with the differential coefficients of these functions with respect to $u, v, \ldots$ and the latter with the differential coefficients of the same functions with respect to $x, y, \ldots$, the quotient with its sign changed obtained by dividing the first of these determinants by the second is, as is well known, the value of the function ∇.

" Putting for shortness

$$\frac{dx}{du} = \alpha, \ \frac{dy}{du} = \beta, \ \cdots \ \frac{dx}{dv} = \alpha', \ \frac{dy}{dv} = \beta', \cdots \text{\&c.}$$

and

$$\frac{du}{dx} = \mathrm{A}, \ \frac{du}{dy} = \mathrm{B}, \ \cdots \ \frac{dv}{dx} = \mathrm{A}', \ \frac{dv}{dy} = \mathrm{B}', \cdots$$

∇ is the reciprocal of the determinant formed with A, B, .. A′, B′, .. &c. Or it is the determinant formed with $\alpha, \beta, \ldots \alpha', \beta', \ldots$ &c.

" From the first of these forms, i. e. considering ∇ as a function of A, B, ..

$$\frac{d\nabla}{d\mathrm{A}} = -\nabla\alpha, \ \frac{d\nabla}{d\mathrm{B}} = -\nabla\beta, \ \cdots \ \frac{d\nabla}{d\mathrm{A}'} = -\Delta\alpha', \ \frac{d\nabla}{d\mathrm{B}'} = -\nabla\beta'$$

where the quantities $\alpha, \beta, \ldots \alpha', \beta', \ldots$ and A, B, .. A′, B′, .. may be interchanged, provided $-\nabla$ be substituted for ∇.

" Hence

$$\frac{1}{\nabla}d\nabla + \alpha d\mathrm{A} + \beta d\mathrm{B} \cdots + \alpha' d\mathrm{A}' + \beta' d\mathrm{B}' \cdots = 0$$

or reducing by

$$\frac{d\mathrm{A}}{dy} = \frac{d\mathrm{B}}{dx} \cdots \frac{d\mathrm{A}'}{dy} = \frac{d\mathrm{B}'}{dx} \cdots \text{\&c.}$$

this becomes

$$\left.\begin{array}{l}\frac{1}{\nabla}d\nabla + \alpha\left(\frac{dA}{dx}dx+\frac{dB}{dx}dy+\cdot\cdot\right) + \beta\left(\frac{dA}{dy}dx+\frac{dB}{dy}dy+\cdot\cdot\right)\cdot\cdot \\ \qquad +\alpha\left(\frac{dA'}{dx}dx+\frac{dB'}{dx}dy+\cdot\cdot\right)+\beta\left(\frac{dA'}{dy}dx+\frac{dB'}{dy}dy+\cdot\cdot\right)\cdot\cdot\end{array}\right\}=0$$

or reducing

$$\frac{1}{\nabla}d\nabla+\left(\frac{dA}{du}+\frac{dA'}{dv}+\cdot\cdot\right)dx+\left(\frac{dB}{du}+\frac{dB'}{dv}\right)dy+\cdot\cdot=0$$

whence separating the differentials and replacing A, A′, . . B, B′, . . by their values,

$$\begin{array}{l}\frac{1}{\nabla}\frac{d\nabla}{dx}+\frac{d}{du}\cdot\frac{du}{dx}+\frac{d}{dv}\cdot\frac{dv}{dx}+\cdot\cdot=0,\\ \frac{1}{\nabla}\frac{d\nabla}{dy}+\frac{d}{du}\cdot\frac{du}{dy}+\frac{d}{dv}\cdot\frac{dv}{dy}+\cdot\cdot=0,\\ \vdots\end{array}$$

(in which $-\nabla$, u, v, . . x, y, . . may be substituted for ∇, x, y, . . u, v, . .).

“ § 2. Let X, Y, . . be any functions of the variables x, y, . . and assume

$$U=X\frac{du}{dx}+Y\frac{du}{dy}+\cdot\cdot$$

$$V=X\frac{dv}{dx}+Y\frac{dv}{dy}+\cdot\cdot$$

U, V, . . being expressed in terms of u, v, . . Then

$$\begin{array}{l}\frac{dU}{du}+\frac{dV}{dv}+\cdot\cdot=X\left(\frac{d}{du}\cdot\frac{du}{dx}+\frac{d}{dv}\cdot\frac{dv}{dx}+\cdot\cdot\right)+Y\left(\frac{d}{du}\cdot\frac{du}{dy}+\frac{d}{dv}\cdot\frac{dv}{dy}+\cdot\cdot\right)\cdot\cdot\\ \qquad+\left(\frac{dX}{du}\cdot\frac{du}{dx}+\frac{dX}{dv}\cdot\frac{dv}{dx}+\cdot\cdot\right)+\left(\frac{dY}{du}\cdot\frac{du}{dy}+\frac{dY}{dv}\cdot\frac{dv}{dy}+\cdot\cdot\right)\cdot\cdot\end{array}$$

i. e.

$$\nabla\left(\frac{dU}{du}+\frac{dV}{dv}+\cdot\cdot\right)=-\left(X\frac{d\nabla}{dx}+Y\frac{d\nabla}{dy}+\cdot\cdot\right)+\nabla\left(\frac{dX}{dx}+\frac{dY}{dy}+\cdot\cdot\right)$$

Also, whatever be the value of M,

$$U\frac{dM\nabla}{du}+V\frac{dM\nabla}{dv}+\cdot\cdot=X\frac{dM\nabla}{dx}+Y\frac{dM\nabla}{dy}+\cdot\cdot$$

And from these two properties

$$\frac{dM\nabla U}{du}+\frac{dM\nabla V}{dv}+\cdot\cdot=\nabla\left(\frac{dMX}{dx}+\frac{dMY}{dy}+\cdot\cdot\right)$$

" § 3. Consider the system of differential equations

$$dx : dy : dz \cdots = X : Y : Z \cdots$$

(where, for greater clearness, an additional letter z has been introduced). From these we deduce the equivalent system

$$du : dv : dw \cdots = U : V : W \cdots$$

Suppose that u and v continue to represent arbitrary functions of x, y, z, but that the remaining function $w, \ldots$ is such as to satisfy $W = 0, \ldots$ (so that $w, \ldots$ may be considered as the constants introduced by obtaining all the integrals but one of the system of differential equations in $x, y, z \ldots$), we have

$$\frac{dM\nabla U}{du} + \frac{dM\nabla V}{dv} = \nabla\left(\frac{dMX}{dx} + \frac{dMY}{dy} + \frac{dMZ}{dz} + \cdots\right)$$

Also the only one of the transformed equations which remains to be integrated is

$$du : dv = U : V, \text{ or } Vdu - Udv = 0,$$

(in which it is supposed that U and V are expressed by means of the other integrals in terms of u and v).

" Suppose M can be so determined that

$$\frac{dMX}{dx} + \frac{dMY}{dy} + \frac{dMZ}{dz} + \cdots = 0$$

(M is what Jacobi terms the multiplier of the proposed system of differential equations), then

$$\frac{dM\nabla U}{du} + \frac{dM\nabla V}{dv} = 0$$

or $M\nabla$ is the multiplier of $Vdu - Udv = 0$, so that

$$\int M\nabla(Vdu - Udv) = \text{const.}$$

" Hence the theorem :—' Given a multiplier of the system of equations $dx : dy : dz \ldots = X : Y : Z \ldots$ (the meaning of the term being defined as above), then if all the integrals but one of this system are known, the remaining integral depends upon a quadrature.' "